SECOND PUBLIC EXAMINATION

Honour School of Natural Science (Physics) Part C: 4 Year Course

Honour School of Physics and Philosophy Part C

Mock Paper C2: Laser Science and Quantum Information Processing

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Answer **four** *questions.*

Start the answer to each question on a fresh page.

The numbers in the margin indicate the weight which the Examiners anticipate assigning to each part of the question.

Do NOT turn over until told that you may do so.

1. The PetaWatt laser VULCAN at Rutherford Laboratories is used to generate very high intensities on a small target. Since 1 PW = 10^{15} W, and the beam can be focused to spot sizes of the order 1 μm in diameter, the electric fields at the focus can be significantly larger than those binding the electrons in an atom.

Assume the VULCAN beam approximates a Gaussian beam, which is a solution to the paraxial wave equation. The (scalar) field (analytic signal) for such a beam is

$$E(r,z,t) = E_0 e^{\mathrm{i}(kz-\omega t)} \frac{e^{\frac{\mathrm{i}kr^2}{2q}}}{q},$$

where z is a longitudinal coordinate, $k = \frac{2\pi}{\lambda}$ the wavenumber, ω the frequency, r the coordinate transverse to z and q a complex beam parameter, of the form $q(z) = z - ib$.

State the relationship between r and $|q|$ that ensures the paraxial approximation is valid and derive expressions for the size of the beam (i.e. the beam diameter) $w(z)$ and the radius of curvature of the wavefront $R(z)$ at an arbitrary position z. [5]

The Gaussian beam is focused to a spot by placing a lens of focal length f at its waist and letting the beam propagate a distance L to a second beam waist. Write down the ray transfer ($ABCD$) matrices for the lens and for the free space propagation, and, using the formula relating the q-parameters at the input and output of a paraxial optical system or otherwise, determine the value of L (in terms of the input beam waist and the focal length of the lens). Under what condition is the beam waist at the focal plane of the lens? [7]

Show that under this condition, the size of the beam waist is approximately the same as the spot size predicted by scalar diffraction theory for a plane wave propagating through a circular aperture of the same size as the input beam. [5]

Determine how big the lens aperture should be in order to allow through 90% of the beam power. If a lens of this aperture is used, what effects will it have on the shape and size of the beam at the focus, and why? [4]

Why might you be forced to choose a lens of such restricting aperture when designing a beam delivery system for VULCAN? In what way could these problems be fixed without choosing a larger aperture lens? [4]

2. Explain briefly why certain media can generate the second harmonic from a monochromatic applied electromagnetic wave. Why does second-harmonic generation take place in only some categories of material? [4]

Describe an experimental arrangement which could be used to demonstrate the generation of the second harmonic of a visible or near-infrared frequency. [5]

What is meant by phase matching? Explain why phase matching is necessary in order to achieve a high conversion efficiency. Show that the phase-matching angle for second-harmonic generation in a negative uniaxial birefringent crystal is given by

$$\sin^2\phi = \frac{(\frac{1}{n'_{\mathrm{o}}})^2 - (\frac{1}{n''_{\mathrm{o}}})^2}{(\frac{1}{n''_{\mathrm{e}}})^2 - (\frac{1}{n''_{\mathrm{o}}})^2}$$

where n'_{o} is the ordinary refractive index at the fundamental frequency, and $n''_{\mathrm{o}}, n''_{\mathrm{e}}$ are the ordinary and extraordinary refractive indices at the second-harmonic frequency. Obtain the angle ϕ for second-harmonic generation of 1.2 μm radiation in KDP using the numerical data below. [10]

Describe what is meant by *optical parametric down conversion.* Discuss the fundamental differences between this process and second harmonic generation. [6]

[The refractive indices of KDP are as follows:

$$\begin{aligned} \lambda &= 1{\cdot}2\ \mu\mathrm{m}: \quad n'_{\mathrm{o}} = 1{\cdot}49017 \quad n_{\mathrm{e}} = 1{\cdot}45884 \\ \lambda &= 0{\cdot}6\ \mu\mathrm{m}: \quad n'_{\mathrm{o}} = 1{\cdot}50927 \quad n_{\mathrm{e}} = 1{\cdot}46827] \end{aligned}$$

3. What is meant by the term *Q-switching*? Include in your discussion sketches of the gain, loss and photon flux as functions of time for a Q-switched laser, as well as a description of the elements needed for such a laser, and a sketch of the cavity layout. [6]

The dynamical equations for the intracavity photon flux and the inversion are

$$\begin{aligned}\frac{\mathrm{d}N^*}{\mathrm{d}t} &= -\frac{\sigma c}{V} N^* n,\\ \frac{\mathrm{d}n}{\mathrm{d}t} &= \frac{1}{\tau_\mathrm{c}}\left(\frac{N^*}{N^*_\mathrm{th}} - 1\right) n,\end{aligned}$$

where N^* is the population inversion density, n the intracavity photon number, τ_c the cavity decay time, and N^*_th the inversion density that would be required to reach threshold for continuous wave operation when the cavity is switched to its high-Q state.

Using these equations show that the intracavity photon density at the peak of the pulse is

$$n = V\left(N^*_\mathrm{i} - N^*_\mathrm{th} - N^*_\mathrm{th}\ln(N^*_\mathrm{i}/N^*_\mathrm{th})\right),$$

and that the output pulse energy is

$$E = \hbar\omega\eta N^*_\mathrm{i} V,$$

where η, the extraction efficiency, is defined by

$$\eta = \frac{(N^*_\mathrm{i} - N^*_\mathrm{f})}{N^*_\mathrm{i}}.$$

The inversion densities before and after the pulse is emitted are N^*_i and N^*_f, the output photon energy is $\hbar\omega$, and V is the mode volume in the gain medium (which is taken to fill the cavity). Explain in simple physical terms the relationship between the pulse energy, the photon energy and the inversion for the case when $N^*_\mathrm{i} \gg N^*_\mathrm{f}$. [10]

Write down an expression for the cavity decay time in terms of the cavity loss after Q-switching and the cavity length, and derive an approximate expression for the pulse duration of the Q-switched laser in terms of the cavity decay time and the other laser parameters. Explain physically the conditions under which the output pulse duration of a Q-switched laser tends to the cavity decay time. [5]

A laser cavity consists of two mirrors, having reflectivities $R_1 = 0{\cdot}80$ and $R_2 = 0{\cdot}98$. The cavity length is $0{\cdot}3\,\mathrm{m}$. The saturation fluence of the gain medium, $\hbar\omega/2\sigma$, is $1\,\mathrm{J\,cm^{-2}}$ and the beam diameter throughout the gain medium is approximately $1\,\mathrm{mm}$. Calculate the energy, peak power and pulse duration from this laser when Q-switched with an initial inversion density 500 times above threshold. [4]

4. Explain what is meant by *homogeneous* and *inhomogeneous* line broadening and give two examples of each class. [5]

In a laser medium with upper level 2 and lower level 1 the normalized lineshape function is $g(\omega)$ and ω_0 is the centre angular frequency of the laser transition. Show that, under conditions in which saturation effects may be neglected, the gain coefficient $\alpha(\omega)$ is given by

$$\alpha(\omega) = \frac{\pi^2 c^2}{\omega_0^2} A_{21}\, g(\omega) \left(N_2 - \frac{g_2}{g_1} N_1 \right) ,$$

where the population densities of the two levels are N_1 and N_2, and their statistical weights are g_1 and g_2. You may assume that

$$g_1 B_{12} = g_2 B_{21}, \quad \text{and} \quad A_{21} = \frac{\hbar {\omega_0}^3}{\pi^2 c^3} B_{21},$$

where B_{12}, B_{21} and A_{21} are the Einstein coefficients of the transition. [8]

The lifetime of the upper level of the 1064 nm laser transition in neodymium-doped yttrium aluminium garnet (Nd:YAG) is 260 μs, the Einstein A-coefficient is $1800\,\mathrm{s}^{-1}$, and the linewidth is 195 GHz. State the dominant broadening mechanism for this transition and estimate its peak optical gain cross-section. You may ignore any effects of the refractive index of the crystal host.

A Nd:YAG laser rod, 50 mm long and of diameter 4 mm, is placed in a two-mirror optical cavity with mirror reflectivities of 100% and 90%. The laser rod is optically pumped by the output from a semiconductor laser operating at 809 nm. Estimate the power of the pump laser required to reach the threshold condition for continuous-wave oscillation on the Nd:YAG laser transition at 1064 nm. [6]

Discuss how the power and spectrum of the laser output behaves as the pump power of the laser rod is increased above the threshold value. How would this behaviour differ if the host for the Nd^{3+} ions were glass rather than a crystalline material? [6]

5. Show that a general pure state of a single qubit can be written as

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$$

Find the corresponding density matrix and give a geometric interpretation of θ and ϕ. [5]

Consider the pure state

$$|\psi^{\perp}\rangle = \cos(\theta'/2)|0\rangle + \sin(\theta'/2)e^{i\phi'}|1\rangle$$

where $\theta' = \pi - \theta$ and $\phi' = \pi + \phi$. Show that $|\psi^{\perp}\rangle$ is orthogonal to $|\psi\rangle$, and give a geometric interpretation of the relationship between $|\psi\rangle$ and $|\psi^{\perp}\rangle$. Use state fidelities to show that $|\psi^{\perp}\rangle$ is the state most unlike $|\psi\rangle$, and that the fidelity between the maximally mixed state and any pure state is 1/2. Show that the state obtained by applying a 180°_{y} rotation to $|\psi^{\perp}\rangle$ has a density matrix which is the transpose of $|\psi\rangle\langle\psi|$. [10]

The operation which converts $|\psi\rangle$ to $|\psi^{\perp}\rangle$ is sometimes called the U-NOT gate, and it can be shown that this gate cannot be implemented physically. It can, however, be approximated. One possible approach is to apply a conventional NOT gate, and another is to measure the qubit in the computational basis and then prepare a qubit in the appropriate state. Find the fidelity between each of these gates and an ideal U-NOT gate for the three initial states corresponding to the intersections of the x, y and z-axes with the Bloch sphere. Comment on the *average* and *worst case* fidelities of these two approximate approaches. [10]

(The fidelity of a gate for a given input is defined as the state fidelity between the output of the gate and the desired output.)

6. What are the differences between *physically* and *mathematically* secure cryptographic protocols? What is the key distribution problem and how can it be solved by quantum cryptography? [8]

Explain the BB84 protocol in detail and draw a schematic optical setup for realizing it. How can an eavesdropper using the intercept/resend strategy be detected? [10]

The communication channel used by Alice and Bob is capable of generating a secret key at a rate of 10 kbit/s if no eavesdropper can be present (i.e. Alice and Bob do not need to publicly compare parts of their key). By what fraction does this rate go down if an eavesdropper using the intercept/resend strategy on each third qubit is to be detected with 99.9% probability within two seconds? [7]

7. State the CHSH inequalities for measurements on two qubits and briefly describe the first Aspect experiments testing these inequalities. Describe two loopholes of these first experiments and explain how they have been addressed [by Aspect and others] in later experiments. [10]

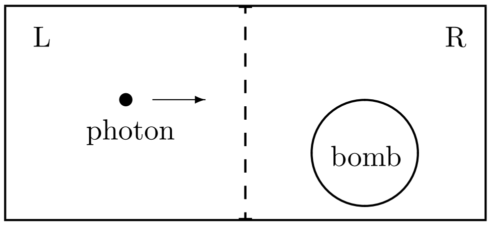
Assume a perfect experimental setup with a photon source that provides GHZ states of the form $|\mathrm{GHZ}\rangle = (|010\rangle + |101\rangle)/\sqrt{2}$. What are the possible measurement outcomes if each qubit is measured in its computational basis? How are these measurement results correlated with each other?

Next rewrite $|\mathrm{GHZ}\rangle$ using the $|\pm\rangle$ basis (or X-basis) for the first qubit and the Y-basis for the second and third qubit. What are the possible measurement outcomes and their correlations if the first qubit is measured in the X-basis and the second and third qubits are measured in the Y-basis?

Finally rewrite $|\mathrm{GHZ}\rangle$ using the X-basis for all three qubits. What are the possible measurement outcomes and their correlations if all qubits are measured in the X-basis? [10]

Explain how measurements on GHZ states in different bases are contradicting local realism. [5]

8. The apparatus shown below is a hypothetical example of an *interaction free measurement.*



The apparatus is built using a long thin box of total length d with perfectly reflecting walls, which is divided into two by an almost perfectly reflecting mirror. The left hand side of the box (L) initially contains a single photon, while the right hand side (R) contains either an active bomb, triggered by a perfect single photon detector, or a passive bomb with no detector. The aim of the experiment is to determine whether the bomb is active or passive without setting it off if it is active.

Begin by ignoring the bomb, and consider a single photon traveling as indicated by the arrow (you may assume that the photon only moves in the left–right dimension). The position of the photon can be written in the $\{|L\rangle, |R\rangle\}$ basis. Assume that the interaction with the central mirror can be described by the evolution operator

$$\begin{pmatrix} \sqrt{1-f^2} & if \\ if & \sqrt{1-f^2} \end{pmatrix}$$

where f depends on the reflectivity of this mirror. State how f relates to p, the probability that a single photon is transmitted by the mirror. Show that the behaviour of the photon can be approximated by continuous evolution under the Hamiltonian $\omega\sigma_x/2$ and determine how the value of ω depends on p and d. Describe the effect of evolution under this Hamiltonian, and find the first time τ at which the photon is certain to be in the right hand side of the box. [10]

Now place a bomb in the right hand side of the box. If a photon detector is attached this can be treated as measuring the position of the photon in the $(|L\rangle, |R\rangle)$ basis. Show that in this case the photon is unlikely to leave the left hand side of the box if p is small (the quantum Zeno effect). [8]

Consider the case of a bomb on the right hand side of a box of length $d = 3\,\text{m}$ with a mirror which reflects 99.99% of incident photons. Calculate the first time at which a photon will certainly be found on the right hand side if the bomb is passive, and the probability that an active bomb will explode during this time. [7]

[End of paper]