In[1]:=
```
σx = {{0, 1}, {1, 0}};
σy = {{0, -I}, {I, 0}};
σz = {{1, 0}, {0, -1}};
Conj[X_] := X /. Y_Complex → Conjugate[Y]
Adjoint[M_] := Conj[Transpose[M]]
```

In[6]:=
```
U[θ_, ϕ_] := MatrixExp[-I * θ * (σx * Cos[ϕ] + σy * Sin[ϕ]) / 2]
```

In[7]:=
```
V[θ_, ϕ_] := U[θ (1 + ϵ), ϕ]
```

In[8]:=
```
Uπ = U[π, 0]
```

Out[8]= {{0, -𝕚}, {-𝕚, 0}}

In[9]:=
```
Vn = V[π, 0]
```

Out[9]= $\left\{\left\{-\mathrm{Sin}\left[\frac{\pi \epsilon}{2}\right], -i\,\mathrm{Cos}\left[\frac{\pi \epsilon}{2}\right]\right\}, \left\{-i\,\mathrm{Cos}\left[\frac{\pi \epsilon}{2}\right], -\mathrm{Sin}\left[\frac{\pi \epsilon}{2}\right]\right\}\right\}$

In[10]:=
```
Fn = Simplify[Tr[Adjoint[Uπ].Vn] / 2]
```

Out[10]= $\mathrm{Cos}\left[\frac{\pi \epsilon}{2}\right]$

In[11]:=
```
Series[Fn, {ϵ, 0, 3}]
```

Out[11]= $1 - \frac{\pi^2 \epsilon^2}{8} + O[\epsilon]^4$

In[12]:=
```
ψ0 = {{1}, {0}}
```

Out[12]= {{1}, {0}}

In[13]:=
```
ABS2[X_] := Conj[X] * X
```

In[14]:=
```
Pn = ABS2[Tr[Adjoint[ψ0].Adjoint[Uπ].Vn.ψ0]]
```

Out[14]= $\mathrm{Cos}\left[\frac{\pi \epsilon}{2}\right]^2$

In[15]:=
```
Series[Pn, {ϵ, 0, 3}]
```

Out[15]= $1 - \frac{\pi^2 \epsilon^2}{4} + O[\epsilon]^4$

In[16]:=
```
VL = Simplify[V[π / 2, 0].V[π, π / 2].V[π / 2, 0]]
```

Out[16]= $\left\{\left\{\mathrm{Sin}\left[\frac{\pi \epsilon}{2}\right]^2, -\mathrm{Cos}\left[\frac{\pi \epsilon}{2}\right] + \frac{1}{2} i\,\mathrm{Sin}[\pi \epsilon]\right\}, \left\{\mathrm{Cos}\left[\frac{\pi \epsilon}{2}\right] + \frac{1}{2} i\,\mathrm{Sin}[\pi \epsilon], \mathrm{Sin}\left[\frac{\pi \epsilon}{2}\right]^2\right\}\right\}$

In[17]:=
```
PL = Simplify[ABS2[Tr[Adjoint[ψ0].Adjoint[Uπ].VL.ψ0]]]
```

Out[17]= $\mathrm{Cos}\left[\frac{\pi \epsilon}{2}\right]^2 + \frac{1}{4} \mathrm{Sin}[\pi \epsilon]^2$

In[18]:=
```
Series[PL, {ϵ, 0, 5}]
```

Out[18]= $1 - \frac{\pi^4 \epsilon^4}{16} + O[\epsilon]^6$

In[19]:=
```
(VL /. ϵ → 0) == Uπ
```

Out[19]= False

In[20]:=
```
(VL /. ϵ → 0) == U[π, π / 2]
```

Out[20]= True

In[21]:=
```
FL = Simplify[Tr[Adjoint[U[π, π / 2]].VL] / 2]
```

Out[21]= $\mathrm{Cos}\left[\frac{\pi\,\epsilon}{2}\right]$

In[22]:=
```
VT = Simplify[V[π, π / 3].V[π, 5 π / 3].V[π, π / 3]]
```

Out[22]= $\left\{\left\{\mathrm{Cos}\left[\frac{1}{2}\pi\,(1+\epsilon)\right]^3,\ -\frac{1}{4}\mathrm{Cos}\left[\frac{\pi\,\epsilon}{2}\right]\left(5\,i+\sqrt{3}-\left(i+\sqrt{3}\right)\mathrm{Cos}[\pi\,\epsilon]\right)\right\},\ \left\{\frac{1}{4}\mathrm{Cos}\left[\frac{\pi\,\epsilon}{2}\right]\left(-5\,i+\sqrt{3}-\left(-i+\sqrt{3}\right)\mathrm{Cos}[\pi\,\epsilon]\right),\ \mathrm{Cos}\left[\frac{1}{2}\pi\,(1+\epsilon)\right]^3\right\}\right\}$

In[23]:=
```
(VT /. ϵ → 0) == Uπ
```

Out[23]= True

In[24]:=
```
FT = Simplify[Tr[Adjoint[Uπ].VT] / 2]
```

Out[24]= $-\frac{1}{4}\mathrm{Cos}\left[\frac{\pi\,\epsilon}{2}\right](-5+\mathrm{Cos}[\pi\,\epsilon])$

In[25]:=
```
Series[FT, {ϵ, 0, 5}]
```

Out[25]= $1-\frac{3\,\pi^4\,\epsilon^4}{128}+O[\epsilon]^6$

In[26]:=
```
FullSimplify[U[π, ϕ4].U[π, ϕ3].U[π, ϕ2].U[π, ϕ1]]
```

Out[26]= $\left\{\left\{e^{i\,(\phi 1-\phi 2+\phi 3-\phi 4)},\ 0\right\},\ \left\{0,\ e^{-i\,(\phi 1-\phi 2+\phi 3-\phi 4)}\right\}\right\}$

In[27]:=
```
MatrixExp[-I * 2 * (ϕ1 - ϕ2 + ϕ3 - ϕ4) * σz / 2]
```

Out[27]= $\left\{\left\{e^{-i\,(\phi 1-\phi 2+\phi 3-\phi 4)},\ 0\right\},\ \left\{0,\ e^{i\,(\phi 1-\phi 2+\phi 3-\phi 4)}\right\}\right\}$

In[28]:=
```
W[ϕ1_, ϕ2_] := V[π, ϕ1].V[2 π, ϕ2].V[π, ϕ1]
```

In[29]:=
```
Simplify[W[ϕ1, ϕ2]] /. ϵ → 0
```

Out[29]= $\{\{1,\ 0\},\ \{0,\ 1\}\}$

In[30]:=
```
Simplify[Normal[Series[Simplify[W[ϕ1, ϕ2]], {ϵ, 0, 1}]] - IdentityMatrix[2]]
```

Out[30]= $\{\{0,\ -\pi\,\epsilon\,(i\,\mathrm{Cos}[\phi 1]+\mathrm{Sin}[\phi 1])\,(1+\mathrm{Cos}[\phi 1]\,\mathrm{Cos}[\phi 2]-i\,\mathrm{Sin}[\phi 1-\phi 2]+\mathrm{Sin}[\phi 1]\,\mathrm{Sin}[\phi 2])\},\ \{\pi\,\epsilon\,(-i\,\mathrm{Cos}[\phi 1]+\mathrm{Sin}[\phi 1])\,(1+\mathrm{Cos}[\phi 1]\,\mathrm{Cos}[\phi 2]+i\,\mathrm{Sin}[\phi 1-\phi 2]+\mathrm{Sin}[\phi 1]\,\mathrm{Sin}[\phi 2]),\ 0\}\}$

In[31]:=
```
Simplify[
 % == Simplify[-ϵ * I * π * ((Cos[ϕ1] + Cos[2 ϕ1 - ϕ2]) * σx + (Sin[ϕ1] + Sin[2 ϕ1 - ϕ2]) * σy)]]
```

Out[31]= True

In[32]:=
```
FullSimplify[Normal[Series[FullSimplify[W[ϕ1, 3 ϕ1]], {ϵ, 0, 1}]] - IdentityMatrix[2]]
```

Out[32]= $\{\{0,\ -2\,i\,\pi\,\epsilon\,\mathrm{Cos}[\phi 1]\},\ \{-2\,i\,\pi\,\epsilon\,\mathrm{Cos}[\phi 1],\ 0\}\}$

In[33]:=
```
Series[U[ϵ * 4 π * Cos[ϕ1], 0], {ϵ, 0, 1}]
```

Out[33]= $\left\{\left\{1+O[\epsilon]^2,\ -2\,i\,\pi\,\mathrm{Cos}[\phi 1]\,\epsilon+O[\epsilon]^2\right\},\ \left\{-2\,i\,\pi\,\mathrm{Cos}[\phi 1]\,\epsilon+O[\epsilon]^2,\ 1+O[\epsilon]^2\right\}\right\}$

In[34]:= `VB = V[π, 0].FullSimplify[W[ArcCos[-1 / 4], 3 * ArcCos[-1 / 4]]]`

Out[34]=
$$\left\{\left\{-\frac{1}{16}\,(15+\text{Cos}[2\,\pi\,\epsilon])\,\text{Sin}\left[\frac{\pi\,\epsilon}{2}\right]-\frac{\mathbb{i}\,\text{Cos}\left[\frac{\pi\,\epsilon}{2}\right]\left(15+7\,\mathbb{i}\,\sqrt{15}+\left(7-\mathbb{i}\,\sqrt{15}\right)\text{Cos}[\pi\,\epsilon]\right)\text{Sin}[\pi\,\epsilon]}{-44\,\mathbb{i}+12\,\sqrt{15}},\right.\right.$$
$$-\frac{1}{16}\,\mathbb{i}\,\text{Cos}\left[\frac{\pi\,\epsilon}{2}\right](15+\text{Cos}[2\,\pi\,\epsilon])-$$
$$\left.\left(\left(105+17\,\mathbb{i}\,\sqrt{15}+\left(-17+7\,\mathbb{i}\,\sqrt{15}\right)\text{Cos}[\pi\,\epsilon]\right)\text{Sin}\left[\frac{\pi\,\epsilon}{2}\right]\text{Sin}[\pi\,\epsilon]\right)\Big/\left(-176\,\mathbb{i}+48\,\sqrt{15}\right)\right\},$$
$$\left\{-\frac{1}{16}\,\mathbb{i}\,\text{Cos}\left[\frac{\pi\,\epsilon}{2}\right](15+\text{Cos}[2\,\pi\,\epsilon])-\frac{\left(15+7\,\mathbb{i}\,\sqrt{15}+\left(7-\mathbb{i}\,\sqrt{15}\right)\text{Cos}[\pi\,\epsilon]\right)\text{Sin}\left[\frac{\pi\,\epsilon}{2}\right]\text{Sin}[\pi\,\epsilon]}{-44\,\mathbb{i}+12\,\sqrt{15}},\right.$$
$$-\frac{1}{16}\,(15+\text{Cos}[2\,\pi\,\epsilon])\,\text{Sin}\left[\frac{\pi\,\epsilon}{2}\right]-$$
$$\left.\left.\left(\mathbb{i}\,\text{Cos}\left[\frac{\pi\,\epsilon}{2}\right]\left(105+17\,\mathbb{i}\,\sqrt{15}+\left(-17+7\,\mathbb{i}\,\sqrt{15}\right)\text{Cos}[\pi\,\epsilon]\right)\text{Sin}[\pi\,\epsilon]\right)\Big/\left(-176\,\mathbb{i}+48\,\sqrt{15}\right)\right\}\right\}$$

In[35]:= `FB = Simplify[Tr[Adjoint[Uπ].VB] / 2]`

Out[35]=
$$\frac{1}{128}\left(150\,\text{Cos}\left[\frac{\pi\,\epsilon}{2}\right]-25\,\text{Cos}\left[\frac{3\,\pi\,\epsilon}{2}\right]+3\,\text{Cos}\left[\frac{5\,\pi\,\epsilon}{2}\right]\right)$$

In[36]:= `Series[FB, {ϵ, 0, 7}]`

Out[36]=
$$1-\frac{5\,\pi^6\,\epsilon^6}{1024}+O[\epsilon]^8$$