Chapter 8

Exercises

8.1 Single qubits

1. Show that if $|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$ then

$$|\psi\rangle\langle\psi| = \tfrac{1}{2}\left(\sigma_0 + s_x\,\sigma_x + s_y\,\sigma_y + s_z\,\sigma_z\right)$$

   where σ_α are the usual Pauli matrices, with α equal to x, y, z or 0. Show that $\mathbf{s} = (s_x, s_y, s_z)$ (the Bloch vector) has unit length, and so $|\psi\rangle\langle\psi|$ can be represented by a point on the unit sphere (Bloch sphere). Show that any mixed state of a single qubit can be written as a point *in* the Bloch sphere. What point does $\frac{1}{2}\sigma_0$ correspond to?

2. Show that $\sigma_\alpha^2 = \sigma_0$, and hence use a series expansion to show that $\exp(-i\theta\,\sigma_\alpha/2) = \cos(\theta/2)\sigma_0 - i\sin(\theta/2)\sigma_\alpha$ without diagonalizing any matrices.

3. Using matrix propagators show that the Hadamard gate can be implemented as 90_y180_x (where rotations are written from left to right; note that propagators must be applied from right to left). Show that other possible implementations include 180_x90_{-y}, $90_{-y}180_z$ and 180_z90_y.

4. Spin echoes. Show that $\phi_z\,180_x\,\phi_z \equiv 180_x$ and thus show that $\phi_z\,180_x\,2\phi_z\,180_x\,\phi_z$ is equivalent to the identity. Similarly show that $\phi_z\,180_x\,\phi_z\,180_x$ and $180_x\,\phi_z\,180_x\,\phi_z$ are also equivalent to the identity. What about $180_y\,\phi_z\,180_y\,\phi_z$? What about $180_x\,\phi_z\,180_y\,\phi_z$? Try to avoid just multiplying matrices mindlessly, but instead reuse partial results and use known properties of propagators where possible.

5. We have used matrices to show that $\mathrm{H}\sigma_z\mathrm{H} = \sigma_x$; now show that $\mathrm{H}\sigma_x\mathrm{H} = \sigma_z$ without multiplying matrices.

6. Another way to do this is to note that H is equal to $(\sigma_x + \sigma_z)/\sqrt{2}$; use this and the known properties of products of Pauli matrices to prove that $\mathrm{H}\sigma_z\mathrm{H} = \sigma_x$.

7. This approach allows some quite complex calculations; to keep life simple it is often better to write X for σ_x etc. Use this approach to show that the product of operators $90_{-y}90_x90_y$ is equivalent to 90_z. Recall that $\mathrm{XY} = i\mathrm{Z}$.

8.2 Physical Systems

1. The Innsbruck ion trap quantum computer is based on the electric dipole forbidden transition between the $D_{5/2}$ excited state and the $S_{1/2}$ ground state of $^{40}Ca^+$ ions (^{40}Ca has nuclear spin $I = 0$, and so there is no hyperfine structure to worry about; this transition is weakly allowed by coupling to the atom's electric quadrupole moment, and has a natural lifetime of about 1 s). This transition can be driven directly at a wavelength of 729 nm. Calculate the limiting spatial resolution in this system (you may assume the Abbe limit), and comment on the expected excited state population at 300 K.

2. In an experiment to observe Rabi oscillations in this system, the population of the $D_{5/2}$ state was found to increase, then decrease, then increase again, with a minimum observed after about 1 μs. Since this is a quadrupole transition, we can't really analyze it using the methods in this book, but let's ignore that. Suppose an electric dipole transition was driven at the same rate: make a reasonable estimate of the electric field strength required. For the strength of the dipole moment you may take $z \sim a_0$.

3. Now calculate the spontaneous decay time of a strongly allowed transition at the same wavelength, which is given by $1/\Gamma = (3\pi\epsilon_0\hbar c^3)/(\omega^3 e^2 z^2)$, and comment on the result.

4. Suppose we tried to excite this transition by brute force, using a very large jump in a static electric field. Estimate the field strength required to make this work, and comment on your result.

5. A typical modern NMR spectrometer has a main magnetic field strength of about 12 T, resulting in a ^1H Larmor frequency of about 500 MHz, while an RF pulse causing a 90° rotation will typically last around 6 μs. Calculate the strength of the *oscillating* magnetic field component of the RF field.

6. Calculate the energy gap between the two spin states of a ^1H in the system discussed above. Assuming a Boltzmann distribution between the two energy states, what are the probabilities of finding a given nucleus in the two states at a temperature of 300 K? Suppose an NMR sample contains 0.2 ml of water at 300 K: what is the excess number of spins in the lower energy state? What temperature is required to place 99% of the spins in the lower energy state?

7. As implied above, a typical NMR sample is a moderately large object (several mm in each direction), containing many identical copies of the same spin. If the magnetic field is different at each spin then the Larmor frequency will also vary, giving rise to *inhomogeneous broadening*. Suppose the natural NMR linewidth is around 1 Hz, which is reasonable: how much variation in the field can we tolerate? Is this practical?

8. In section 5.3 we explored some of the properties of birefringent wave plates, with a particular emphasis on quarter wave plates ($\phi = \pi/2$). Now evaluate the unitary transformation performed by a half wave plate, and show how this can be used to implement NOT gates and Hadamard gates directly.

9. Suppose I make a beam of vertically polarized light, and pass it through an ideal piece of polaroid film with a vertical axis. The light beam will be completely transmitted. Now suppose I put a second polarizer after the first one, at an angle θ; the transmitted fraction will drop to $\cos^2\theta$, with no transmission occurring at $90°$ (the Law of Malus). Now suppose I use two ideal polarizers after the first one, at angles of $45°$ and $90°$: what will be the transmitted fraction in this case? Now suppose I use a sequence of n polarizers, equally spaced up to $90°$ (so that for the case $n = 3$ the first polarizer is at $0°$ and the next three are at $30°$, $60°$ and $90°$ respectively). What is the transmission for general values of n? How about $n = 90$? What is the value in the limit $n \to \infty$?

8.3 Two qubits

1. Show that a controlled-NOT gate can be built out of Hadamard gates and a controlled-σ_z gate without using explicit matrices in your argument.

2. Use the "bitwise addition modulo 2" description of the controlled-NOT gate to show that a network of three controlled-NOT gates will swap the values of two qubits in eigenstates. Hence show that this network acts as a SWAP get for any separable state of two qubits.

3. Calculate an explicit matrix form for the SWAP gate. What does this gate do to a pair of qubits in a Bell state? Why is this answer not surprising?

4. The Ising Hamiltonian, which plays a key role in many proposed implementations of quantum computing, takes the form $\mathcal{H} = (2\pi\nu/2)\,\sigma_z \otimes \sigma_z$. Show that a combination of a period of evolution under the Ising Hamiltonian for a time $t = 1/(4\nu)$ and a bilateral 90_{-z} rotation is equivalent to the controlled-σ_z gate (ignoring global phases).

5. Suppose Alice and Bob share an entangled pair of qubits in the state ψ^-. Find local operations that Bob can use to convert this to the other three Bell states.

6. It can be shown that any single qubit gate can be constructed out of a suitable network of Hadamard gates and $\mathrm{T} = \sqrt{\mathrm{S}} = \sqrt[4]{\mathrm{Z}}$ gates. Use this fact to prove that the singlet state ψ^- is unaffected by any bilateral unitary operation.

7. A pure state is said to be separable (and therefore not entangled) if it can be written as a direct product of single qubit states; a mixed state is said to be separable (and therefore not entangled) if it can be written as a mixture of separable pure states. Now suppose that Alice and Bob start with a pair of qubits in the separable state $|0\rangle|0\rangle$, and that they try to create an entangled state by LOCC. Inspired by the standard network, Alice applies a Hadamard to her qubit and then measures it; if she gets a $|0\rangle$ she does nothing, but if she gets a $|1\rangle$ she tells Bob to apply a NOT gate to his qubit. Find the resulting state, and show that it is not entangled. What is the state fidelity between the resulting state and each of the four Bell states? Can you describe the resulting state as a mixture of Bell states?