1 Quantum information: exercises and problems

These exercises are closely based on those in the forthcoming book by Jones and Jaksch, and the numbering of sections follows the order in that book. In a few cases the exercises have been slightly rewritten so they stand alone. *These exercises are intended for self study and worked answers are provided on the course website; you are very strongly advised to make a serious attempt at all these exercises rather than just looking up the answer!* If anything is still unclear ask your class tutor; it is probably best to let him know before the class.

The last section contains the three *problems* which will form the bulk of the class material; two are taken from finals papers and one from the mock paper. Links to these three problems are available on the course website; worked answers are *not* provided.

1. Quantum bits and quantum gates

1. Show that if $|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$ then

$$|\psi\rangle\langle\psi| = \tfrac{1}{2}\left(\sigma_0 + s_x\,\sigma_x + s_y\,\sigma_y + s_z\,\sigma_z\right).$$

   Show that $\mathbf{s} = (s_x, s_y, s_z)$ (the Bloch vector) has unit length, and so $|\psi\rangle\langle\psi|$ can be represented by a point on the unit sphere (Bloch sphere).
2. Show that any mixed state of a single qubit can be written as a point *in* the Bloch sphere. What point does $\frac{1}{2}\sigma_0$ correspond to?
3. Show that $\sigma_\alpha^2 = \sigma_0$, where σ_α are the usual Pauli matrices, with α equal to x, y, or z. Hence use a series expansion to show that $\exp(-i\theta\,\sigma_\alpha/2) = \cos(\theta/2)\sigma_0 - i\sin(\theta/2)\sigma_\alpha$ without diagonalizing any matrices.
4. Using matrix propagators show that the Hadamard gate can be implemented as $90^\circ_y\,180^\circ_x$ (where rotations are written from left to right; note that propagators must be applied from right to left). Show that other possible implementations include $180^\circ_x\,90^\circ_{-y}$, $90^\circ_{-y}\,180^\circ_z$, and $180^\circ_z\,90^\circ_y$.
5. We have used matrices to show that HZH = X; now show that HXH = Z without multiplying matrices.
6. Rewrite the general state of a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in the X-basis (that is as a superposition of $|+\rangle$ and $|-\rangle$). Show that the result of an X-measurement on this state is identical to the effect of applying a Hadamard gate, performing a Z-measurement, and then applying another Hadamard gate.
7. Explain *why* the result of the previous question works, and why any single qubit measurement gate can be achieved by combining unitary transformations with a Z-measurement.

2. An atom in a laser field

1. Explain why the selection rules derived for hydrogen atoms can also be applied to ions with a single electron in their outer shell, such as Ca^+.
2. Consider the possibility of using the $^2S_{1/2}$ and $^2P_{1/2}$ to encode a qubit in $^{40}Ca^+$ ions. For simplicity we assume that the matrix element $\langle z \rangle = \langle \psi_i | z | \psi_f \rangle \sim a_0$ for allowed transitions, and is zero for forbidden transitions. Calculate the spontaneous decay time of this transition using $1/\Gamma = (3\pi\epsilon_0 \hbar c^3)/(\omega^3 e^2 \langle z \rangle^2)$, and estimate the electric field strength needed to perform Rabi flopping on this transition using a resonant oscillating electric field.
3. Suppose we tried to excite this transition by brute force, using a very large jump in a static electric field. Estimate the field strength required to make this work, and comment on your result.
4. Estimate the limiting spatial resolution in this system (you may assume the Abbe limit).
5. Comment on the expected excited state population at 300 K.
6. The peak electric field in a laser beam can be calculated using $E_p = 2\sqrt{Pc\mu_0/A}$, where P is the power of the laser and A is the cross sectional area of the beam. Estimate the laser power required to perform Rabi flopping assuming the laser beam is focused to a uniform spot with a diameter given by the Abbe limit.

3. Spins in magnetic fields

1. A typical modern NMR spectrometer has a main magnetic field strength of about 12 T, resulting in a ^1H Larmor frequency of about 500 MHz, while an RF pulse causing a 90° rotation will typically last around 6 μs. Calculate the strength of the *oscillating* magnetic field component of the RF field.
2. Calculate the energy gap between the two spin states of a ^1H in the system discussed above. Assuming a Boltzmann distribution between the two energy states, what are the probabilities of finding a given nucleus in the two states at a temperature of 300 K?
3. Suppose an NMR sample contains 0.2 ml of water at 300 K: what is the excess number of spins in the lower energy state? What temperature is required to place 99% of the spins in the lower energy state?
4. As implied above, a typical NMR sample is a moderately large object (several mm in each direction), containing many identical copies of the same spin. If the magnetic field is different at each spin then the Larmor frequency will also vary, giving rise to *inhomogeneous broadening*. Suppose the natural NMR linewidth is around 1 Hz, which is reasonable: how much variation in the field can we tolerate? Is this practical?
5. There are many different sequences which can be classified as spin echoes, differing only in fine details. Confirm that $\phi_z\, 180_x\, \phi_z \equiv 180_x$, and show that $\phi_z\, 180_x\, 2\phi_z\, 180_x\, \phi_z$ is equivalent to the identity. Similarly show that $\phi_z\, 180_x\, \phi_z\, 180_x$ and $180_x\, \phi_z\, 180_x\, \phi_z$ are also equivalent to the identity. What about $180_y\, \phi_z\, 180_y\, \phi_z$ and $180_x\, \phi_z\, 180_y\, \phi_z$?

4. Photon techniques

1. We have already explored some of the properties of birefringent wave plates, with a particular emphasis on quarter wave plates ($\phi = \pi/2$). Now evaluate the unitary transformation performed by a half wave plate, and show how this can be used to implement NOT gates and Hadamard gates directly.
2. Show that the coherent state $|\alpha\rangle$ is correctly normalised. Find as a function of $|\alpha|$ the fraction of laser pulses containing at least one photon, and the fraction of such pulses containing exactly one photon. Hence confirm the results for $\alpha = \sqrt{0.1}$ given in the text.

5. Two qubits and beyond

1. Show that a controlled-NOT gate can be built out of Hadamard gates and a controlled-σ_z gate without using explicit matrices in your argument.
2. Use the "bitwise addition modulo 2" description of the controlled-NOT gate to show that a network of three controlled-NOT gates will swap the values of two qubits in eigenstates. Hence show that this network acts as a SWAP gate for any separable state of two qubits.
3. Calculate an explicit matrix form for the SWAP gate. What does this gate do to a pair of qubits in a Bell state? Why is this answer not surprising?
4. Find explicit expressions for the four computational basis states of a two qubit system in terms of superpositions of the four Bell states.
5. Show that the entangling network shown in equation **??** can be used to produce all four Bell states by using different initial states for the input qubits.
6. How can the four Bell states be converted into four distinguishable states in the computational basis?

6. Measurement and entanglement

1. Suppose I make a beam of vertically polarized light, and pass it through an ideal piece of polaroid film with a vertical axis. The light beam will be completely transmitted. Now suppose I put a second polarizer after the first one, at an angle θ; the transmitted fraction will drop to $\cos^2\theta$, with no transmission occurring at 90° (the Law of Malus). Now suppose I use two ideal polarizers after the first one, at angles of 45° and 90°: what will be the transmitted fraction in this case? Now suppose I use a sequence of n polarizers, equally spaced up to 90° (so that for the case $n = 3$ the first polarizer is at 0° and the next three are at 30°, 60° and 90° respectively). What is the transmission for general values of n? What is the value in the limit $n \to \infty$?
2. Suppose Alice and Bob share an entangled pair of qubits in the state $|\psi^-\rangle$. Find local operations that Bob can use to convert this to the other three Bell states.
3. It can be shown that any single qubit gate can be constructed out of a suitable network of Hadamard gates and $\mathrm{T} = \sqrt{\mathrm{S}} = \sqrt[4]{\mathrm{Z}}$ gates. Use this fact to prove that the singlet state $|\psi^-\rangle$ is unaffected by any bilateral unitary operation.
4. If two qubits in the Bell state $|\psi^-\rangle$ are measured in the computational basis they will always disagree. Use the result of the previous exercise to show that the same property

holds for $|\psi^-\rangle$ if the two qubits are measured in the *same* basis, whatever basis is chosen. Does this work for the other three Bell states?

5. A pure state is said to be separable (and therefore not entangled) if it can be written as a direct product of single qubit states; a mixed state is said to be separable (and therefore not entangled) if it can be written as a mixture of separable pure states. Now suppose that Alice and Bob start with a pair of qubits in the separable state $|0\rangle|0\rangle$, and that they try to create an entangled state by LOCC. Inspired by the standard network, Alice applies a Hadamard to her qubit and then measures it; if she gets a $|0\rangle$ she does nothing, but if she gets a $|1\rangle$ she tells Bob to apply a NOT gate to his qubit. Find the resulting state, and show that it is not entangled. What is the state fidelity between the resulting state and each of the four Bell states? Can you describe the resulting state as a mixture of Bell states?

Problems

Please answer the following finals questions from C2 papers

1. Mock paper question 5
2. 2007 paper question 5
3. 2008 paper question 5

and hand these in for marking as arranged by your class tutor. Links to these questions can be fond on the course website. Other questions you should be able to attempt at this stage are 2005 Q5, 2009 Q5, and 2010 Q5 (except the last part).