Lecture 7: Optics / C2: Quantum Information and Laser Science: Guided Waves

January 19, 2009

1 Optical fibers and wave guiding

We have seen that electromagnetic waves in free space (or homogeneous media) spread by diffraction. However, they can be recollimated by lens or mirrors in such a way as to be self-reproducing after same number of periods of any given optical arrangement satisfying certain conditions. So we can have non-spreading beams, after a fashion. One can imagine taking such optical relays further, by making the lengths of free-space propagation very small, and interspersing them with weak lenses. This requires the ability to fabricate these structures. The waves that propagate in them are called guided waves, and are part of a large class of propagating waves associated with inhomogeneous media. Among the most useful in this class are waves in optical fibres, strands of glass structured to support non-spreading wave propagation over many km of material, and with the ability for non-straight-line transmission. These are used extensively in optical communication systems (see Brooker Ch. 14). The main feature of all such structure is an inhomogeneous distribution of dielectrics such that a medium of high delectric permittivity is sandwiched between two of lower permittivity, as shown in the figure below.

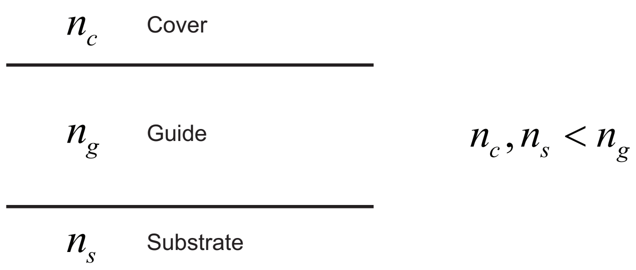

Figure 1:

In this case electromagnetic energy tends to concentrate itself in the guiding layer, since this is the lowest energy configuration. Another way to understand the guiding properties of such a structure is in terms of ray optics: in this picture the light entering the guiding layer tends to experience total internal reflection at the interfaces provided the input ray angle is not too large. This causes the ray to bounce along the channel, remaining confined in the guide layer, even when there are small bends in the structure.
This simple idea can be used to understand the basic features of an optical fiber. A typical optical fiber used for telecommunications has a cross-section as shown in the figure.
Typical materials are glasses doped with Germanium: GeO_2, SiO_2. These form the higher-index core, and are surrounded by less-doped glasses that form the cladding layer.
For rays entering the fiber in the plane A-B in the above figure, the ray propagation is sketched below:
We can assess the angles at which light will emerge from the fiber based on the above model. The smallest value of θ_i is at the critical angle for total internal reflection

$$\theta_{i\ \min} = \sin^{-1}(n_2/n_1) \tag{1}$$

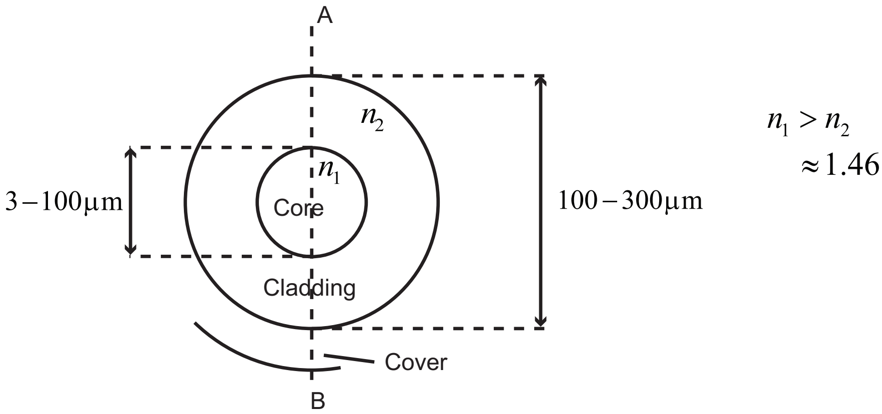

Figure 2:

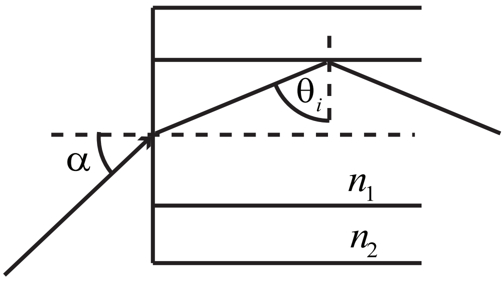

Figure 3:

In this case α has its maximum value:

$$\begin{aligned} 1 \times \sin\alpha_m &= n_1 \sin(90 - \theta_{i\ \min}) = n_1 \cos\theta_{i\ \min} \\ \sin\alpha_m &= n_1\sqrt{1 - \sin^2\theta_{i\ \min}} \\ \sin\alpha_m &= \sqrt{n_1^2 - n_2^2} \end{aligned} \tag{2}$$

In general it is reasonable to take $(n_1 - n_2)/n_2 = \Delta \ll 1$, so

$$\begin{aligned} \sqrt{n_1^2 - n_2^2} &= \sqrt{(n_1 - n_2)(n_1 + n_2)} \\ &= \sqrt{\frac{n_1 - n_2}{n_1} n_1(n_1 + n_2)} \\ &\approx \sqrt{\Delta 2 n_1^2} \\ &\approx n_1\sqrt{2\Delta} \end{aligned} \tag{3}$$

A typical value for the normalised index difference is $\Delta = 0.009$. Using a reasonable value for the refractive index of glass gives for the numerical aperture of the fiber:

$$\text{N.A.} = \sin^2\alpha_m \approx 0.2. \tag{4}$$

Consequently θ_{min} is $\theta_m = 11.5°$.

2 Fiber modes

Just as there are several Gaussian beams that will "fit" in a cavity - the Hermite-Gaussian modes - there are several modes, or field patterns that can exist in a fiber. To find these we must solve the appropriate

wave equations.
For a step-index fiber, we use the geometry as shown in the following figure, in which the cladding is taken to extend to infinity. Therefore we solve the scalar wave equations in cylindrical coordinates, taking boundary conditions that the field amplitude dies to zero at infinity. To make the problem simpler, we will assume that the field is nearly linearly polarized so we are considering a class of modes labelled LP_{lm}.

$$\left(\nabla^2 - \frac{n^2}{c^2}\partial_t^2\right)\vec{E} = 0 \tag{5}$$

or

$$\left\{\frac{1}{r}\partial_r\left(r\partial_r\vec{E}\right) + \frac{1}{r^2}\partial_\phi^2\vec{E} + \partial_z^2\vec{E} - \frac{n^2}{c^2}\partial_t^2\vec{E}\right\} = 0 \tag{6}$$

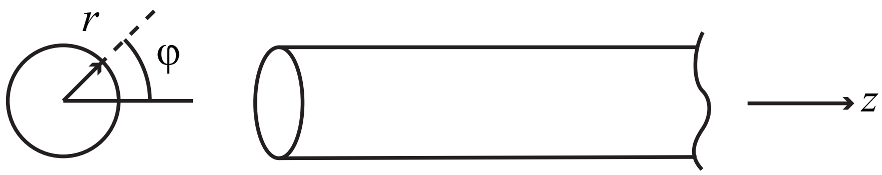

Figure 4:

The fiber geometry suggests that we seek a separable solution of the form

$$\vec{E} = \vec{\epsilon}_x F_1(r) F_2(\phi) e^{i(\beta z - \omega t)}, \tag{7}$$

where β is the propagation constant, giving the wavenumber in the $z-$direction. Substituting this into the wave equation yields

$$\left\{\frac{1}{r}\partial_r\left(r\partial_r F_1\right)F_2 + \frac{1}{r^2}\partial_\phi^2 F_2\, F_1 + \left(-\beta^2 + n^2\frac{\omega^2}{c^2}\right)F_1 F_2\right\} = 0. \tag{8}$$

and taking the simple azimuthal dependence

$$F_2(\phi) = e^{\pm i2\pi l\phi} \tag{9}$$

gives:

$$\frac{1}{r}\partial_r\left(r\partial_r F_1\right) + \left\{\frac{n^2\omega^2}{c^2} - \beta^2 - \frac{l^2}{r^2}\right\}F_1 = 0 \tag{10}$$

Now in the core, radius a, refractive index n_1, we define the transverse wavenumber k_T

$$k_T^2 = \frac{n_1^2\omega^2}{c^2} - \beta^2 = n_1^2 k^2 - \beta^2 \tag{11}$$

Since $k_T^2 > 0$, the maximum value of β is $n_1 k$. Then $F_1(r)$ satisfies

$$\left[r^2\partial_r^2 + r\partial_r + (k_T^2 r^2 - l^2)\right]F_1(r) = 0 \tag{12}$$

This is Bessel's equation, and is satisfied by Bessel functions of the first and second kind. Since those of the second kind (Neumann functions) $N_l(k_T r)$ diverge at the origin $r = 0$, we shall only be interested in the more physically plausible Bessel functions of the first kind $J_l(k_T r)$. In this case therefore we postulate:

$$F_1(r) = A\, J_l(k_T r) \tag{13}$$

in the core. In contrast to this, in the cladding we set

$$k_T^2 = -\gamma^2 = \frac{n_2^2\omega^2}{c^2} - \beta^2 < 0 \tag{14}$$

Since $n_2 < n_1$, then

$$\left[r^2\partial_r^2 + r\partial_r - (\gamma^2 r^2 + l^2)\right] F_1(r) = 0. \tag{15}$$

and $F_1(r)$ is therefore a *modified* Bessel function of the first or second kind. Of this class, those of the first kind, $I_l(\gamma r)$ diverge as $r \to \infty$, so we expect the solution to be one of the second kind:

$$F_1(r) = B\, K_l(\gamma r). \tag{16}$$

These solutions must match at the boundary between the core and the cladding. That is, the components of $\vec{E}$ and $\vec{H}$ tangential to the core-cladding interface must be continuous. So we expect that E_z, H_z, E_ϕ, H_π will be continuous at $\rho = a$.

Now Bessel functions of the first kind oscillate as a function of r, and modified Bessel functions of the second kind tend to decay for large argument. So we may expect a field of the generic form shown in Fig. 5.

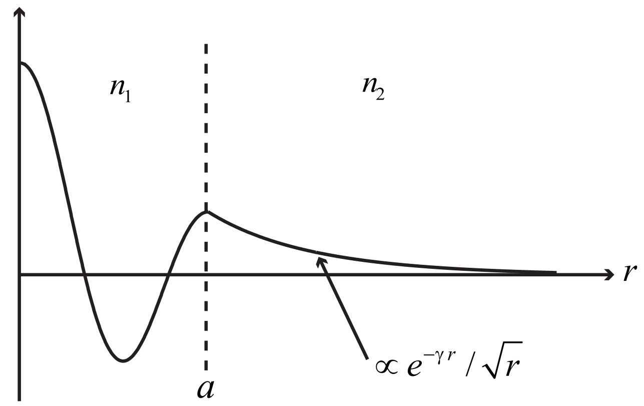

Figure 5: Field pattern in a cylindrical waveguide (e.g. an optical fiber) as a function of the radius. In the core ($r \ll a$) the field is a Bessel function of the first kind. In the cladding ($r \gg a$) it is a modified Bessel function of the second kind. The functions and their derivatives are matched at $r = a$

Using the linear polarization assumption, we may write down the fields in core and cladding:

$$\vec{E}^{\text{core}}(r,\phi) = \vec{\epsilon}_x E_l \frac{J_l(k_T r)}{J_l(k_T a)} \cos(l\phi) e^{i(\beta z - \omega t)} \tag{17}$$

$$\vec{E}^{\text{clad}}(r,\phi) = \vec{\epsilon}_x E_l \frac{K_l(\gamma r)}{K_l(\gamma a)} \cos(l\phi) e^{i(\beta z - \omega t)} \tag{18}$$

As with the Gaussian beams the other non-negligible field vector components can be calculated using Maxwell's equations. The other significant components is H_y, and the other components E_y, E_z, H_x, H_z are small, but non-zero.

In order to get these fields, we assumed that $\vec{E}$ was continuous at the boundary between the two media. In fact we should have that $\vec{D}$ is continuous. But the error in assuming this is small.

$$Error = 1 - \frac{E_y^{clad}(r=a)}{E_y^{core}(r=a)} \tag{19}$$

Since $\vec{D}$ is continuous, we expect

$$n_2^2 \epsilon_0 E_y^{clad}(a) = n_1^2 \epsilon_0 E_y^{core}(a) \tag{20}$$

$$\therefore \frac{E_y^{clad}(a)}{E_y^{core}(a)} = \frac{n_1^2}{n_2^2} \tag{21}$$

and

$$Error = \frac{n_2^2 - n_1^2}{n_2^2} = \frac{(n_2 - n_1)(n_2 + n_1)}{n_2^2} \approx -2\Delta \ll 1 \tag{22}$$

The boundary matching equations lead to another condition on the fields that must be satisfied for propagating waves, that relates the propagation constant β, to the frequency ω (or k). I will not do this calculation in detail, [1] but the procedure is to use Maxwell's 3rd and 4th equations $\nabla \times \vec{E} = i\mu\omega\vec{H}$ and $\nabla \times \vec{B} = i\omega\vec{E}$, to calculate H_y and E_z and E_y and H_z respectively. Then the cylindrical symmetry of the problem requires that the azimuthal fields be matched at the boundary, so one can determine E_ϕ and H_ϕ through the relation

$$\begin{aligned} E_\phi &= \frac{i}{k_T^2}\left[\frac{\beta}{r}\partial_\phi E_z - \omega\mu\partial_r H_z\right] & (23) \\ H_\phi &= \frac{i}{k_T^2}\left[\frac{\beta}{r}\partial_\phi H_z - \omega\epsilon_0 n^2 \partial_r E_z\right] & (24) \end{aligned}$$

Following through this algebra leads to the result

$$-k_T \frac{J_{l-1}(k_T a)}{J_l(k_T a)} = \gamma \frac{K_{l-1}(\gamma a)}{K_l(\gamma a)} \tag{25}$$

which is the LP mode dispersion relation, relating β to $k = \omega/c$. It can only be solved numerically, but yields a non-linear dependence of β on k which means that the fiber has dispersion due to the guiding of waves. This has a non-negligible impact on optical telecommunication systems.

3 Fiber mode functions and electric field distributions

The existence of a mode in the fiber is determined by the cut-off condition for a guided wave. The wave is no longer considered to be guided when $\gamma = 0$, since then the field does not decay further into the cladding and thus begins to experience loss due to diffraction. This detemines the minimum wavelength for which a stable guided wave exists. For a mode of order l, the cut-off condition is thus determined by the zeroes of the function $J_{l-1}(k_T a)$ (since the right-hand side of the dispersion relation Eqn.(25) is zero). Now

$$\begin{aligned} k_T^2 &= n_1^2 \frac{\omega^2}{c^2} - \beta^2 \\ &= \frac{\omega^2}{c^2}\left(n_1^2 - \frac{c^2\beta^2}{\omega^2}\right) & (26) \end{aligned}$$

but also $\gamma^2 = \beta^2 - n_2^2\omega^2/c^2$ so at cut-off, when $\gamma = 0$,

$$c^2\beta^2/\omega^2 = n_2^2 \tag{27}$$

and

$$k_T^2 = \frac{\omega^2}{c^2}(n_1^2 - n_2^2). \tag{28}$$

The argument of the Bessel functions on the left-hand side of Eqn.25

$$k_T a = \frac{2\pi a}{\lambda}\sqrt{(n_1^2 - n_2^2)} \tag{29}$$

is called the $\mathcal{V}$ parameter of the fiber.

$l = 0$ solution

The dispersion relation gives $J_{-1}(k_T a) = 0$. Since $J_{-1}(x) = J_1(x)$ for l odd, the condition for cut-off is $k_T\, a = 0$ (corresponding to the first zero of the Bessel function $J_1(x)$). Now any mode for which k_T is

[1] The details can be found in D. Marcuse Light Transmission Optics, (Van Nostrand Reinhold, New York 1982), Ch. 8

greater than the value needed to satisfy this condition can propagate in the fiber. In this case, there is no minimum wavelength below which light will not propagate as an LP_0 mode. The corollary is that a fiber of any radius can support LP_0 mode.

The field pattern for the LP_0 mode is

$$E_x^{core} = E_0 \frac{J_0(k_T r)}{J_0(k_T a)} e^{i(\beta z - \omega t)} \tag{30}$$

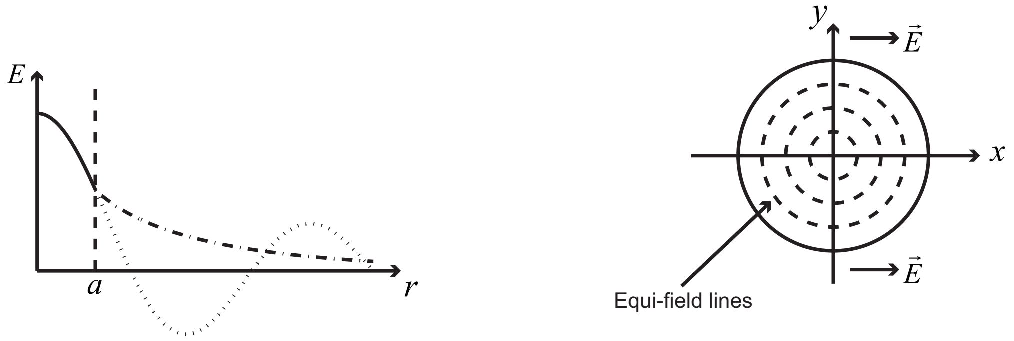

Figure 6:

There is no ϕ-dependence for the $l = 0$ solution, and it is obviously two fold degenerate, since we could have chosen $\vec{E} = \vec{y}E_y$ instead of the x-polarization taken in the example.
Since there are several zeroes for $J_1(k_T a)$ as a function of a, we expect that several different LP_0 modes can propagate if a is large enough. These are labelled by different m values, corresponding to the order of the zero of $J_1(k_T a)$ under consideration so

$$\begin{aligned} J_1(k_T a) = 0 \qquad \text{for} \qquad & k_T a = 0 && m = 1 \\ & k_T a = 3.832 && m = 2 \\ & k_T a = 7.016 && m = 3 \\ & k_T a = 10.174 && m = 4 \\ & \text{etc.} \end{aligned}$$

So, an LP_{01} mode will exist for values of k_T between 0 and $3.832/a$, and an LP_{02} mode will exist for values of k_T between $3.832/a$ and $7.016/a$ Physically the mode numbers represent how many zeroes of the field are contained in the core region of the fiber. For example, in the LP_{01} mode, the field (for $0 \leq k_T a \leq 3.832$) is shown in Fig.7 . At cut-off, the field node occurs exactly at $r = a$.

$l = 1$ **solution**
The cut-off condition is set by the zeroes of $J_0(k_T a)$. The first zero of this is at $k_T a_m = \mathcal{V} = 2.405$, so that a, the fiber radius, has a minimum value to support this kind of mode. Only modes for which

$$\mathcal{V} < 2.405 \tag{31}$$

can propagate. For wavelengths of the order $\lambda = 1.5\mu$m, this implies a core diameter in the region $2\mu\text{m} < a < 5\mu\text{m}$.
The number of field zeroes in the core region is given by the order of the zero of $J_0(k_T a)$:

$$\begin{aligned} J_0(k_T a) = 0 \qquad \text{for} \qquad & k_T a = 2.405 && m = 1 \\ & k_T a = 5.520 && m = 2 \\ & k_T a = 8.654 && m = 3 \end{aligned} \tag{32}$$

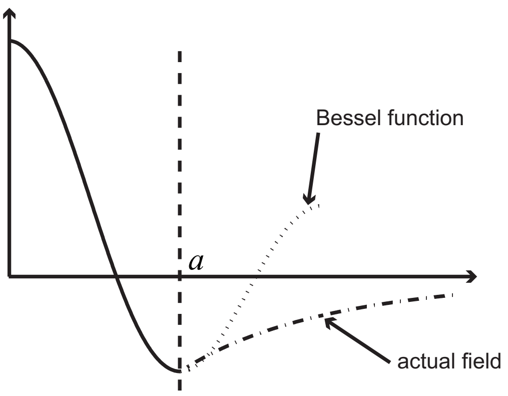

Figure 7:

An example of the field pattern for the LP_{11} mode is

$$E_x^{core} = E_1 \frac{J_1(k_T r)}{J_1(k_T a)} \cos\phi e^{i(\beta z - \omega t)} \tag{33}$$

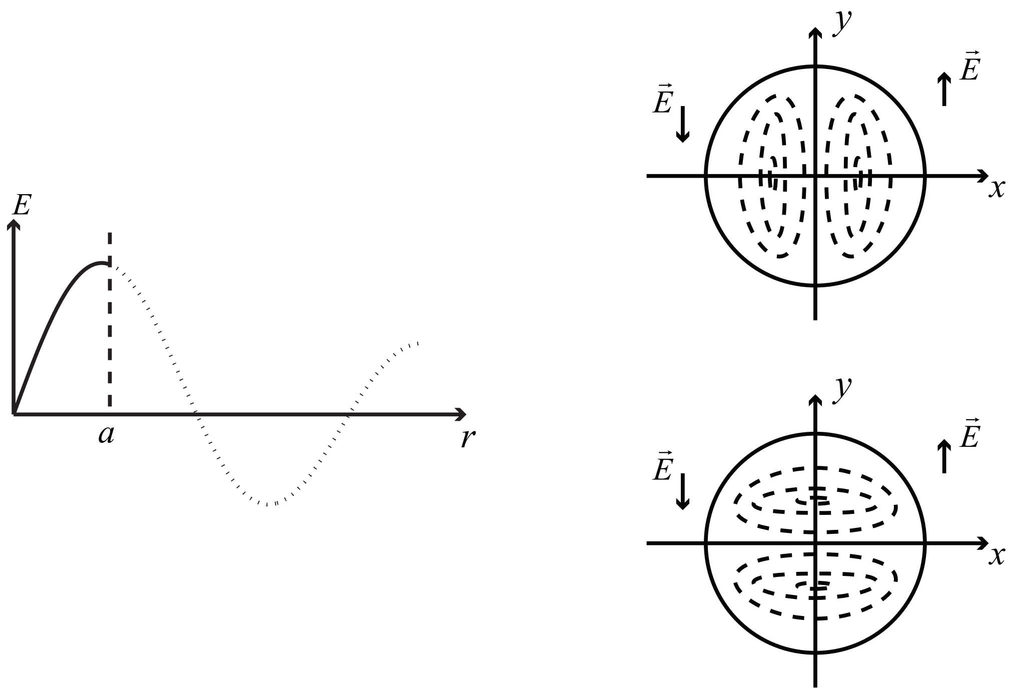

Figure 8:

The two figures show a second type of degeneracy when we replace the $\cos\phi$ by $\sin\phi$. In both cases the field direction changes when $\phi > \pi/2$ so the direction in each of the "lobes" is opposite. There is an overall 4-fold degeneracy (sin, cos, $\hat{x}$, $\hat{y}$).

4 Dispersion

The variation of wavelength with frequency may cause deleterious effects in fibers. In particular any variation of the dependence of β on k beyond a linear one causes pulse broadening. We can determine the size of the nonlinear terms from the dispersion relation. Let

$$\omega = \phi(k) \tag{34}$$

Then

$$\frac{\partial \omega}{\partial k} = \text{group velocity} \tag{35}$$
$$\frac{\partial^2 \omega}{\partial k^2} = \text{group velocity dispersion} \tag{36}$$

As you have seen, in the ultrafast lasers lectures, the latter term gives rise to pulse broadening. Such broadenging can be a problem for optical fiber communications, since the pulses (or absences thereof) that are used to represent the "1"s and "0"s of a bit stream respectively, can be broadened enough that they overlap. It is then difficult to tell the value of the bit and thus the message is corrupted.

In the step index fiber the relevant quantity in the set of derivatives of β, the propagation constant (akin to k above) with respect to k, the wavevector (akin to ω above). We seek $d\beta/dk$ in the first instance. Now the dispersion relation can be written in terms of two parameters

$$\mathcal{V} = \frac{2\pi a}{\lambda}\sqrt{n_1^2 - n_2^2} \tag{37}$$
$$b = \frac{\beta^2/k^2 - n_2^2}{n_1^2 - n_2^2} \tag{38}$$

So pairs of values of $\mathcal{V}$ and b must satisfy:

$$\mathcal{V}\sqrt{1-b}\frac{J_{l-1}(\mathcal{V}\sqrt{1-b})}{J_l(\mathcal{V}\sqrt{1-b})} = -\mathcal{V}\sqrt{b}\frac{K_{l-1}(\mathcal{V}\sqrt{b})}{K_l(\mathcal{V}\sqrt{b})} \tag{39}$$

Here a choice of $\mathcal{V}$ implies k, and a choice of b implies β. Thus:

$$\begin{aligned}\beta^2 &= k^2[n_2^2 + b(n_1^2 - n_2^2)] \\ &= n_2^2 k^2[1 + 2\Delta b]\end{aligned} \tag{40}$$

assuming Δ is small, as we have seen is usually the case. Then expanding the square root gives:

$$\begin{aligned}\beta &\approx n_2 k[1 + b\Delta] \\ &= n_2 k + n_2 k b\Delta\end{aligned} \tag{41}$$

Now the second term in this expansion is, in terms of $\mathcal{V}$

$$n_2 k b\Delta = \frac{b\mathcal{V}}{a}\sqrt{\frac{\Delta}{2}} \tag{42}$$

So that the dispersion relation can be used to find the group velocity:

$$\begin{aligned}\frac{d\beta}{dk} &= \frac{d}{dk}(n_2 k) + \frac{d}{dk}(n_2 b k\Delta) \\ &= \underbrace{\frac{d}{dk}(n_2 k)}_{\text{material dispersion}} + n_2\Delta \underbrace{\frac{d}{d\mathcal{V}}(b\mathcal{V})}_{\text{waveguide dispersion}}\end{aligned} \tag{43}$$

The first of these terms is the familiar one, the second is a new one arising from the confinement of the waves in the transverse dimension. The physical origin of this effect is that the field is spread across two different media, in which the speed of light is different. Therefore the mode sees a sort of "average" of the speeds, depending on how confined it is in the core. So a mode that spreads very little into the cladding moves with speed close to c/n_1, whereas one which resides largely in the cladding moves with speed close to c/n_2. Modes in between compromise between these two extremes. Typical values for the two terms in Eqn. 43 are $d\beta/dk/c = 5$ns/km for materials and $d\beta/dk/c = 0.5$ns/km for waveguide dispersion.

There is also a component of multimode dispersion - the difference in group velocity between modes of higher mode number. This can be quite large ($d\beta/dk/L|_{mn} \approx 50$ns/km) and is due to the path difference between an axis ($l = 0, m = 0$) type rays and off-axis ($l > 0, m > 0$) rays. The physical origin of this effect can be most easily appreciated using the ray model of propagation as shown in Fig.9.

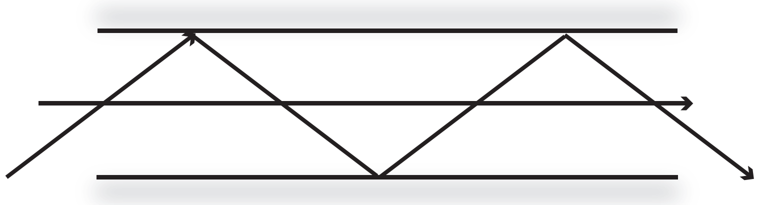

Figure 9: The origin of multimode dispersion in fibers in terms of ray optics. The fundamental mode has $k_T = 0$, and thus its wavevector is directed along the axis of the fiber. Higher order modes have $k_T > 0$, and thus "bounce" from one edge of the core to the other. The overall optical path length for these modes is higher for a given length of fiber than for the fundamental mode.