Lecture 6: Optics / C2: Quantum Information and Laser Science: Coherence

November 19, 2008

1 Random waves

We have discussed up till now monochromatic waves of the form $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, in which the phase evolution is predictable from point to point and from time to time. In most real cases (even with a laser) this is not usually the case. Most real waves are not such idealised forms and have a random amplitude or phase from point to point and time to time. This leads to important consequences in practical measurements and for new applications.
For a single wave, we can write (for example)

$$E_1(\vec{r},t) = \epsilon_1 e^{i(\vec{k}\cdot\vec{r}-\omega t+\phi_1)}, \tag{1}$$

where ϵ_1 and ϕ_1 are both real random variables. Then if we measured the intensity of this beam over an ensemble, described by the probability distribution $\mathbb{P}(\epsilon_1, \phi_1)$, we should find

$$I = |E_1(\vec{r},t)|^2 = \epsilon_1^2, \tag{2}$$

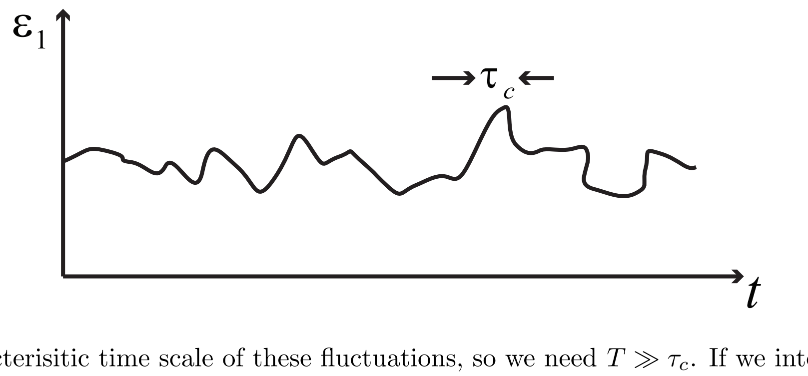
so that the mean intensity is

$$\langle I\rangle = \int d\epsilon_1 d\phi_1 \mathbb{P}(\epsilon_1, \phi_1)\epsilon_1^2. \tag{3}$$

Usually we can make an ergodic hypothesis, which allows us to replace an ensemble average with a time average, so that we would get the same mean intesity by using a "slow" detector

$$\langle I\rangle = \frac{1}{T}\int_{-T/2}^{T/2} dt\ I. \tag{4}$$

Since the field is fluctuating in time:



Now τ_c is the characterisitic time scale of these fluctuations, so we need $T \gg \tau_c$. If we interfere two such beams together, then we find

$$\langle I\rangle = \int d\epsilon_1 d\epsilon_2 d\phi_1 d\phi_2\ \mathbb{P}(\epsilon_1, \epsilon_2, \phi_1, \phi_2)(\epsilon_1^2 + \epsilon_2^2 + 2\epsilon_1\epsilon_2 \cos(\phi_1 - \phi_2)) \tag{5}$$

If the phases vary over a broad enough range, and are statistically independent (*i.e.* $\mathbb{P}(\epsilon_1, \epsilon_2, \phi_1, \phi_2) = \mathbb{P}(\epsilon_1, \phi_1)\mathbb{P}(\epsilon_2, \phi_2)$) then the last term in parentheses average to zero, and we are left with averaging the sum of the two intensities. In this case the waves are said to be mutually incoherent and

$$\langle I\rangle = \langle I_1\rangle + \langle I_2\rangle. \tag{6}$$

and more generally

$$\langle I\rangle = \sum_i \langle I_i\rangle \tag{7}$$

for a set of waves characterized by a parameter $\{i\}$.

2 Characteristic parameters

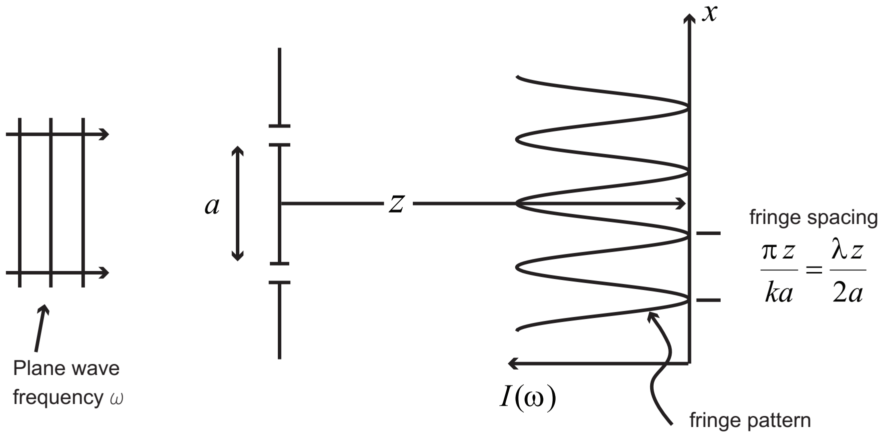
We have assumed in the last section that the amplitude and phase are random variables. Often it suffices to consider simply the phase as such, sinc this have the greatest effect when a set of waves is incident on a detector. In this case $\langle I\rangle = \sum_i \langle I_i\rangle = \int diP(i)I_i$. It is then left to decide the important characteristics that form the set $\{i\}$.
EM waves have 3 degrees of freedom (excluding polarization);, the two components of the transverse wavevector and the frequency ($\vec{k}_\perp$, ω), or equivalently the transverse spatial coordinates and time ($\vec{r}$, t). The fourth variables k_z or z are found from the dispersion relation:

$$\frac{\omega}{c} = |\vec{k}| = \sqrt{\vec{k}_\perp \cdot \vec{k}_\perp + k_z^2} \tag{8}$$

or its time domain equivalent. Therefore we should examine for ensembles that are described by distributions of these parameters $\mathbb{P}(\vec{k}_\perp, \omega)$.

3 Temporal or longitudinal coherence

Let us consider distributions of the form $P(\omega)$ only. That is, the interferometer is illuminated by a broadband source, with random phase between the different spectral components. In order to understand the resulting interference pattern, it is instructive to recall an archetype: the Young's double slit illuminated by a single plane wave.



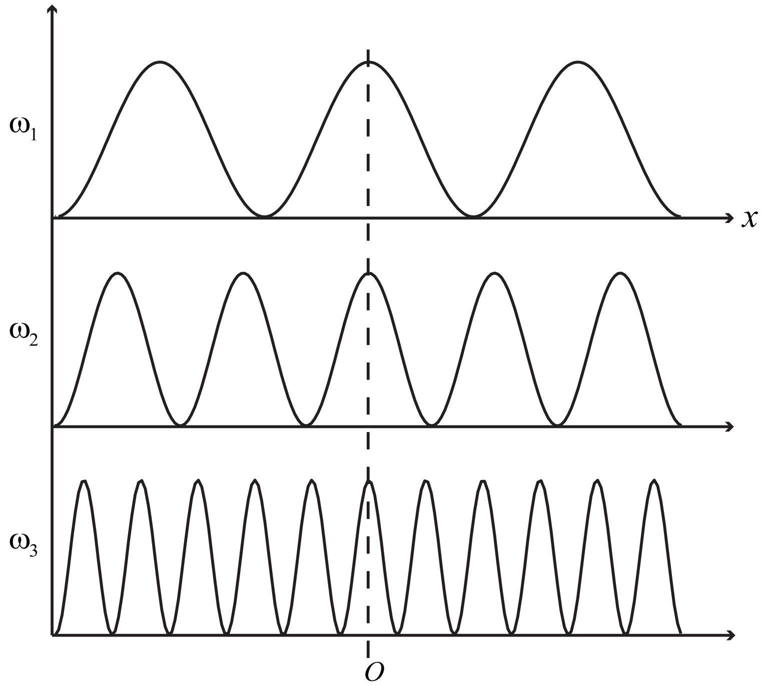
The fringe pattern in the Frauhofer zone, assuming narrow slit-like apertures, is given by:

$$\begin{aligned} I &= I(\omega)\cos^2(k(\omega)ax/z) \\ &= \frac{I(\omega)}{2}\left(1 + \cos\left(\frac{2a\omega x}{cz}\right)\right) \end{aligned} \tag{9}$$

where $I(\omega) = \epsilon_\omega^2$ is the intensity of the incident wave at frequency ω. Now consider the slit being illuminated by light with a range of frequencies, so that $I(\omega)$ describes the distribution of intensities at these frequencies. In this case the interference pattern will be the superpostion of the interference patterns:

$$\langle I \rangle = \int d\omega \, \frac{I(\omega)}{2} \left(1 + \cos\left(\frac{2a\omega x}{cz}\right)\right). \tag{10}$$

What does this look like? Note that the scale of the interference pattern changes with frequency and thus the superposition contains lots of terms where "peaks" of one distribution will overlap "valleys" of another.



The consequence is that, away from the region near $x = 0$, the intensity will consist of an equal number of peaks and valleys, and this will average to a constant value, more or less. Given our formula we can be more explicit about this. The intensity at the screen is

$$\langle I \rangle = \frac{1}{2} \int d\omega \, I(\omega) + \frac{1}{4} \int d\omega I(\omega) e^{i \frac{2ax}{cz} \omega} + \frac{1}{4} \int d\omega I(\omega) e^{-i \frac{2ax}{cz} \omega} \tag{11}$$

Examining the second and third terms, we find that $2ax/cz$ has the dimensions of time. Defining

$$\tau = \frac{2ax}{cz}. \tag{12}$$

Then the second term is just the Fourier transform of the spectrum of the illuminating source;

$$\int d\omega I(\omega) e^{i\omega\tau} = \tilde{I}(\tau) \tag{13}$$

Likewise the third term is the Fourier transform $\tilde{I}(-\tau)$. From a knowledge of Fourier transforms, we know that if $I(\omega) = I^*(\omega)$ and $I(\omega) = I(-\omega)$, then $\tilde{I}(\tau) = \tilde{I}(-\tau)$ and we are left with:

$$\langle I \rangle = \frac{1}{2}(\tilde{I}(0) + \tilde{I}(\tau)). \tag{14}$$

$\tilde{I}(\tau)$ is called the temporal correlation function of the source, and we will examine one or two simple examples.

Consider the case of a Gaussian spectrum:

$$I(\omega) = Ie^{-(\omega-\omega_0)^2/\gamma^2}, \tag{15}$$

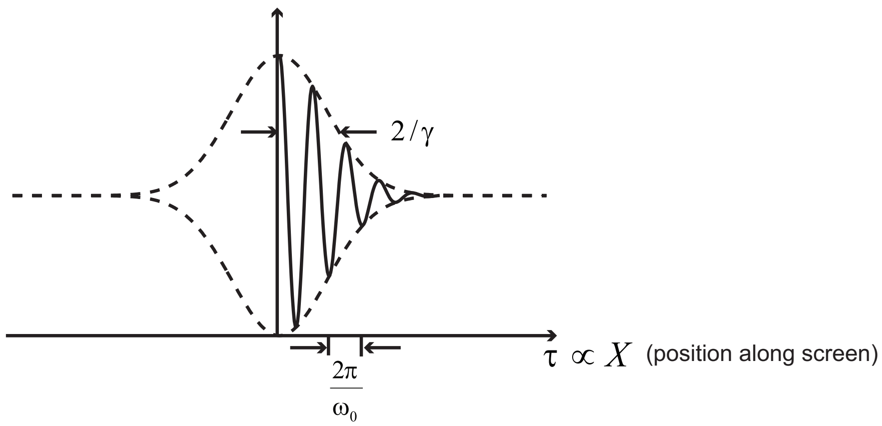
where ω_0 is the mean frequency of the light and γ is the source bandwidth. Inserting this into the expression for the correlation function gives

$$\begin{aligned}\tilde{I}(\tau) &= I\int d\omega e^{-(\omega-\omega_0)^2/\gamma^2} e^{i\omega\tau} \\ &= I\sqrt{\frac{\pi}{1/\gamma^2}} e^{-\frac{\tau^2}{4/\gamma^2}} e^{-i\omega_0\tau} \\ &= I\gamma\sqrt{\pi}e^{-\gamma^2\tau^2/4}e^{-i\omega_0\tau} \end{aligned} \tag{16}$$

$$\therefore \qquad \tilde{I}(\tau) + \tilde{I}(-\tau) = 2I\gamma\sqrt{\pi}e^{-\gamma^2\tau^2/4}\cos\omega_0\tau. \tag{17}$$

Consequently the fringe pattern as a function of τ becomes

$$\langle I\rangle = I\gamma\sqrt{\pi}\left(1 + e^{-\gamma^2\tau^2/4}\cos\omega_0\tau\right). \tag{18}$$



What this shows is that there are interference fringes of high contrast near $\tau = 0$, but that the fringe visibility decreases as $|\tau|$ increases. Once $|\tau|$ is comparable with a few times the inverse bandwidth of the source, the fringes have disappeared into the back ground. This value of τ is called the correlation time of the source.

In order to arrive that this result we have assumed that:

1. Each frequency component has a random phase with respect to the next.
2. The detector averages long enough that all random phases are realized.

Under these conditions, we may add the intensities of individual frequency components and get to the above result.

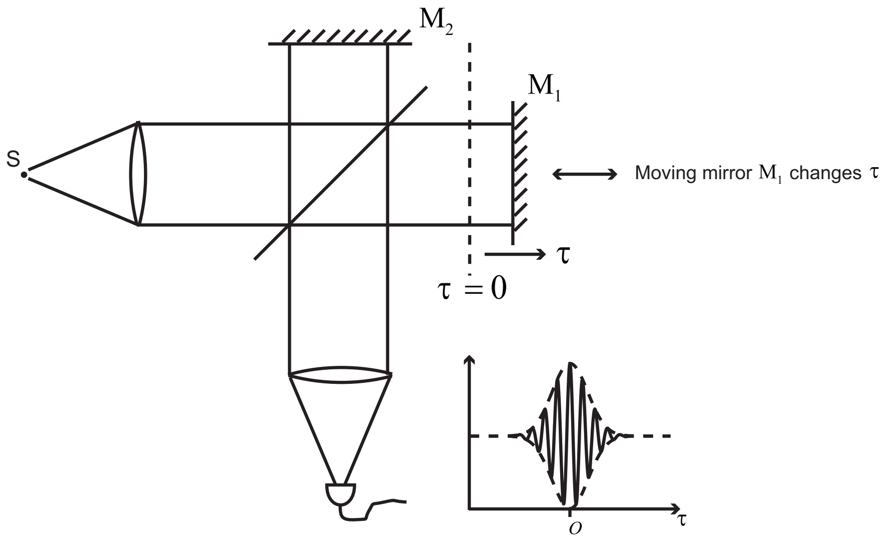
4 Optical coherence tomography (OCT) and Fourier-transform spectrometry (FTS)

Since the visibility of the fringe pattern using a broad band source depends only on the path difference delay τ, it is easy to see how to contruct a Michelson interferometer to measure the pattern:

Scanning one of the mirros while keeping the other fixed changes τ, and the intensity at the detector then maps out the interference pattern:

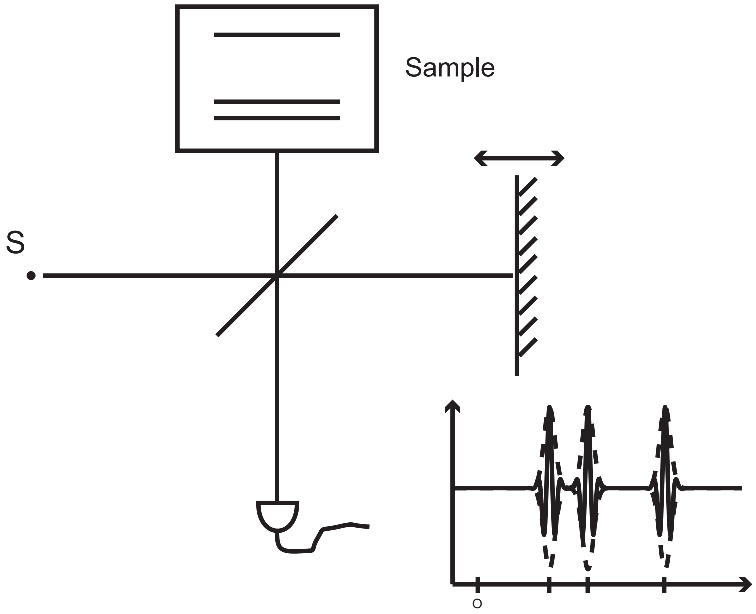
$$\langle I\rangle = \langle I\rangle(\tau) = \frac{1}{2}(\tilde{I}(0) + \mathbb{R}\mathrm{e}\tilde{I}(\tau)). \tag{19}$$

This arrangement has two important applications:



1. Optical Coherence Tomography (OCT). This is a method for measuring the structure of tissue by reflecting light from the layers immediately below the surface and determining the depth of the surface from the location of the peak of the fringe pattern.

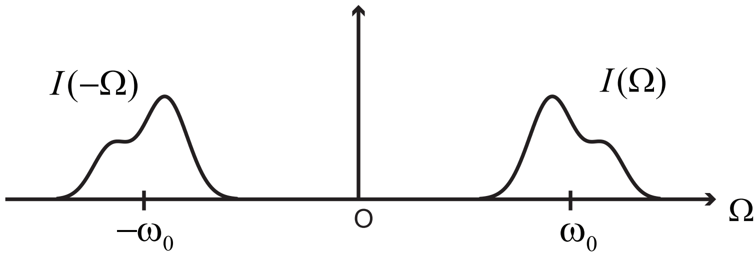
   We have seen that the fringe pattern is peaked at $\tau = 0$. Thus we can locate the mirror position $\tau = 0$ very accurately - if the source has a large enough bandwidth. The precision of the measurement is $\propto 1/\gamma$. Also, say we replace M_2 by a sample with several buried surfaces. Then scanning M_1 gives a peak in the interference pattern whenever $\tau = \tau_n$ where τ_n is related to the depth of the buried surface. This method is used to locate tissue in biological samples, and is especially useful for profiling the eye, say, prior to surgery.



2. Fourier-Transform Spectroscopy (FTS). Since the fringe pattern as a function of τ is related to the Fourier transform of the source spectrum, it can be inverted to obtain the spectrum. In this case the

fringe pattern is recorded, digitized, resampled and inverse transformed using a computer.

$$\begin{aligned}\langle\tilde{I}\rangle &= \int d\Omega\, e^{i\Omega\tau}\langle I\rangle(\tau) \\ &= I(0) + I(\Omega) + I(-\Omega)\end{aligned} \tag{20}$$

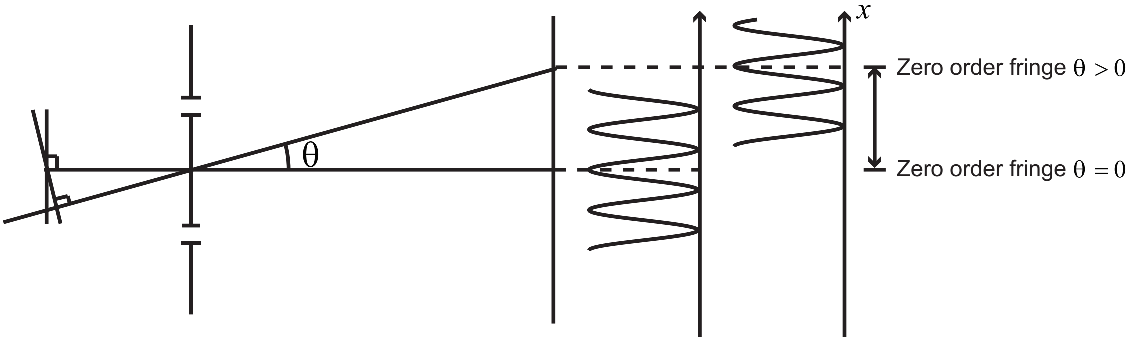


The spectrum can be obtained by digitally filtering one sideband or the other.

This technique finds application in the measurement of spectra in the near infrared where it is hard to find good detector arrays for use with a grating spectrometer.

5 Spatial or transverse coherence

In this case the parameter of interest is the transverse wavevector of the waves. The source is described by the distribution $P(\vec{k}_\perp)$, and all waves have the same frequency. Again there is an random phase between different angular components of the illumination. In this case the physics is different, but still simple: the angle of incidence of the light at the slits affects the lateral position of the fringes, which have the same spacing (which only depends on the wavelength and distance to the screen, given a slit separation).

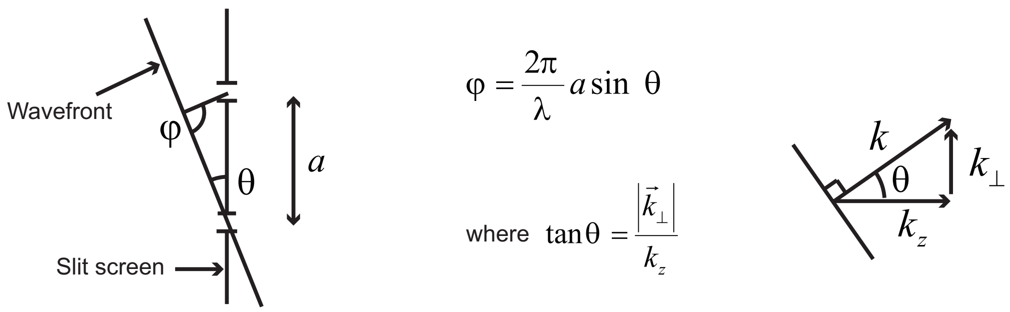


The fringe shift reflects the additional phase of the wave at the upper slit as compared to the lower slit, which depends on the angle of incidence. Then, in the Fraunhofer zone, the interference pattern for light incident at angle θ is

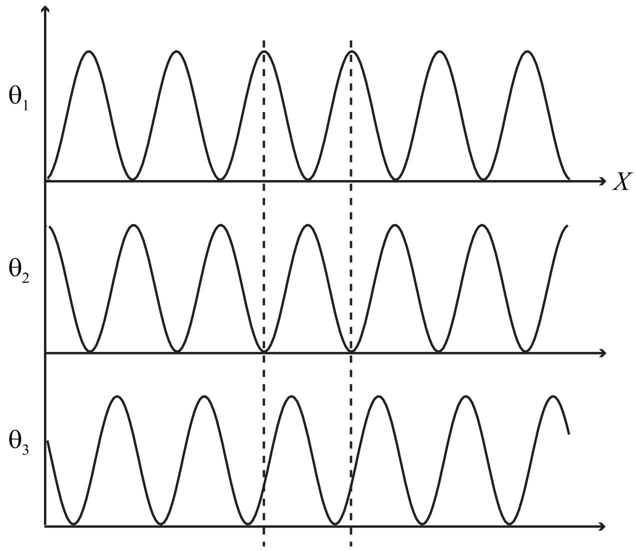
$$I = \frac{1}{2}I(\theta)\left(1 + \cos\left(\frac{2akx}{Z} + ka\sin\theta\right)\right) \tag{21}$$

where $I(\theta)$ is related to the distribution of $\vec{k}_\perp$ (transverse wavevectors) in the source.
The fringes have shifted by δx:

$$\delta x = -\frac{aZ}{2a}\sin\theta = -\frac{Z}{2}\sin\theta \tag{22}$$



Just as with the previous case, it is clear that a distribution of angles of light from the source will give a distribution of patterns of the fringe peaks, meaning some will sit on the valleys of other fringe patterns and the whole interference patterns will gradually wash out.



Using the same procedure as before we can determine the fringe pattern explicitly, given the angular distribution of the source:

$$\langle I \rangle = \frac{1}{2} \int d\theta \, I(\theta) \left(1 + \cos\left(ka\varphi + ka\sin\theta\right)\right) \tag{23}$$

where $\tan\varphi = x/Z$ is the position we are looking at on the screen. If we remain in the paraxial limit, both for the source distribution and for the observation region; we have

$$\langle I \rangle = \frac{1}{2} \int d\theta \, I(\theta) + \frac{1}{4} \int d\theta \, I(\theta) e^{ika\theta} e^{ika\varphi} + \frac{1}{4} \int d\theta \, I(\theta) e^{-ika\theta} e^{-ika\varphi}. \tag{24}$$

or, if $\bar{I}(\xi)$ represents a spatial distribution of the source that is the Fourier transform of the angular distribution $I(\theta)$:

$$\bar{I}(\xi) = \int_{-\infty}^{\infty} d\theta \, I(\theta) e^{i\xi\theta} \tag{25}$$

[Note: Of course θ can only range from $[-\pi, +\pi]$, so the limits of the integral make little sense. But recall that we are remaining in the paraxial limit where $I(\theta)$ is only non zero for a small range of θ near θ_0. In

this case it does not matter if the integral limits have finite range, the actual results will be the same as the full F.T.]
Then

$$\langle I \rangle = \frac{1}{2} \left\{ \bar{I}(0) + \bar{I}(ka)\cos(ka\varphi) \right\}. \tag{26}$$

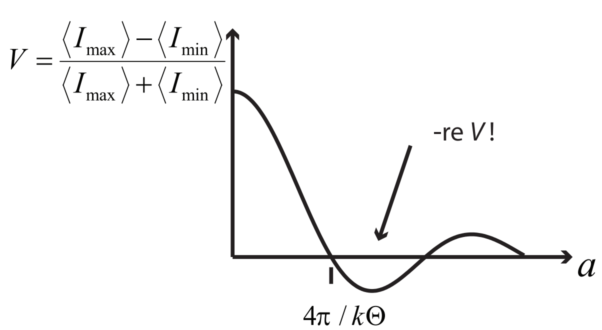
Just as before, the fringe visibility decreases as $\bar{I}(\xi)$ decreases. In this case, we can explore the function $\bar{I}(\xi)$ by varying the slit separation a.
For example, consider a source with an angular distribution of radiation

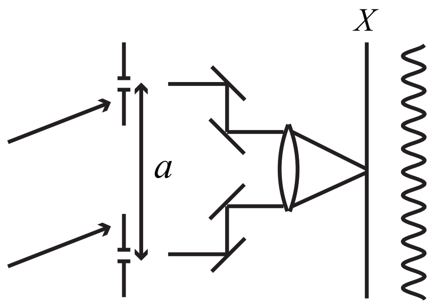
$$I(\theta) = I\mathrm{rect}(\theta/\Theta) \tag{27}$$

Then

$$\bar{I}(ka) = I \int_{-\Theta/2}^{\Theta/2} d\theta\, e^{ika\theta} = I \frac{e^{ika\Theta/2} - e^{-ika\Theta/2}}{ika} = I\Theta\,\mathrm{sinc}(ka\Theta/2) \tag{28}$$

So if we look at the fringe visibility in an experiment where the slit separation is changed we find that it has the form





Just as for the spectral case, the fringe visibility maps out the angular distribtution of the source light. In this case the position of the slits at the first zero of V tells the angular diameter of the source Θ:

$$V = 0 \qquad \text{first at} \qquad a = a_0 = \frac{4\pi}{k\Theta}$$

$$\therefore \qquad \Theta = \frac{4\pi}{ka_0} \tag{29}$$

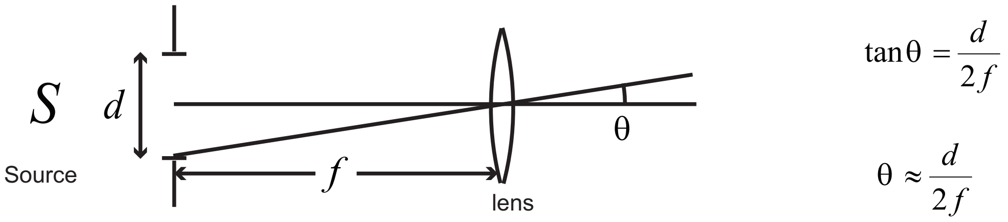
What about negative visibility? This simply means that the position of the dark and bright fringes have swapped - where a bright fringe occured as a was increased from $a = 0$ to $a = a_0$, now occurs a dark fringe.

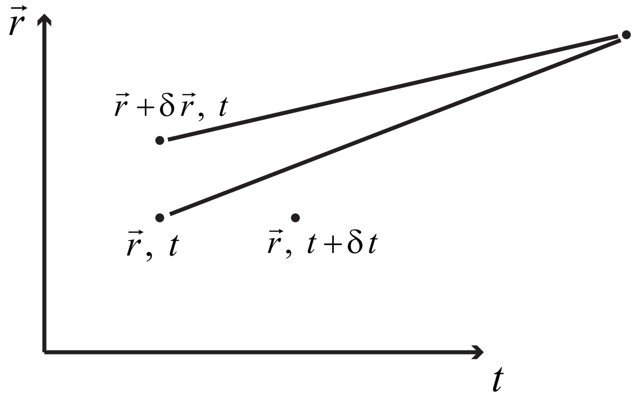
6 The Michelson Stellar Interferometer

This notion is the basis of the Michelson Stellar Interferometer, which was used by Michelson (the second US Nobel Prize winner) to determine the angular diameter of stars. The arrangement he used is similar to the figure shown above.
Finally, in most laboratory experiments, including interference microscopes, the angular range of illumination of the slits is proportional to the physical size of the source itself. Consider the arrangement:
Then, assuming the source S is a *primary, incoherent source* (*i.e.* the light emanating from each part on it has a random phase with respect to the light from all the other parts) the above ideas apply.

7 Coherence and statistical optics

Coherence concerns how the field at different space-time points is related. For example, over what region of space can we predict the phase of the wave given its phase at another location? The first figure below





shows how the field at two different space-time points contributes to the field at a third space-time point. The second figure indicates that if the fields from points 1 and 2 give rise to high-visibility interference fringes at point $\vec{r}, t$, then we may ask whether there will also be good interference at point $\vec{r}, t + \delta t$. This depends on the space-time separation of points 1 and 2. So long as

$$\delta t \quad < \quad \frac{2\pi}{\Delta\omega} \qquad \Delta\omega \text{ width of spectrum} \tag{30}$$

$$\delta\vec{r} \quad < \quad \frac{2\pi}{k\Delta\theta} \qquad \Delta\theta \text{ width of angular spectrum,} \tag{31}$$

there will still be good interference.