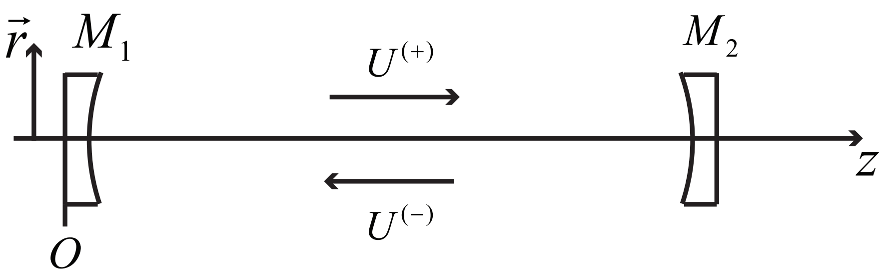
Lecture 5: Optics / C2: Quantum Information and Laser Science: Optical Cavities

November 13, 2008

1 Boundary conditions

The Gaussian beam is a reasonable facsimile of a propagating laser beam that is both directional and monochromatic. We therefore expect that same beam will "fit" inside a laser cavity - meaning that it satisfies the boundary conditions for the electromagnetic field at the mirrors. Let us consider a simple model, using only a scalar field to represent the electric field of the wave. The field inside the cavity has both left and right-going components:

$$U^{(+)} = U(\vec{r}, z)e^{i(kz-\omega t)} \tag{1}$$
$$U^{(-)} = U(\vec{r}, z)e^{i(-kz-\omega t)} \tag{2}$$



Then

$$U^{(+)} + U^{(-)} = |U(\vec{r}, z)|e^{-i\omega t} 2\cos(kz + \phi(z)), \tag{3}$$

where $\phi(z) = \arg U(\vec{r} = 0, z)$ is the on-axis phase. This field satisfies the boundary conditions at the mirrors, which we take to be metal for now (though the concept works too for the more usual dielectric mirror), and to be located at $z = 0$ and $z = L$, provided $U(\vec{r}, z = 0) = U(\vec{r}, z = L) = 0$, *i.e.*:

$$kL + \phi(L) = (2n + 1)\pi/2. \tag{4}$$

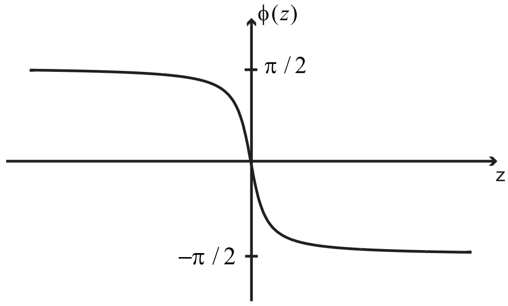
That is the phase must be an odd integer multiple of $\pi/2$, where n is the "mode number". In the simple case when $\phi(L) = 0$, the mode frequency is:

$$k_n L = n\pi + \pi/2 \tag{5}$$
$$\omega_n = n\frac{c\pi}{L} + \frac{c\pi}{2L} \tag{6}$$
$$\nu_n = n\frac{c}{2L} + \frac{c}{4L}. \tag{7}$$

The mode spacing is always $c/2L$, which determines the allowed frequencies in the cavity (and, for ammodelocked laser, the repetition rate of the pulses). It turns out that the mode spacing is independent of $\phi(z)$, although the exact frequencies of the modes do depend on this function. So, what is $\phi(z)$? For the Gaussian beam we have been dealing with, it is the function

$$\phi(z) = \tan^{-1}\frac{z}{b}. \tag{8}$$

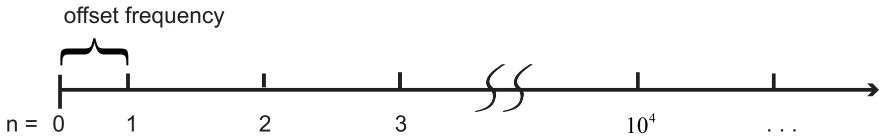
That is, the Guoy phase. But more general Gaussian beams exist. For example the Hermite-Gaussian



functions:

$$U(\vec{r}, z) = U_0 e^{i(kz-\omega t)} \frac{e^{ikr^2/2q(z)}}{q(z)} \underbrace{H_l\left(\frac{x}{w/\sqrt{2}}\right) H_m\left(\frac{y}{w/\sqrt{2}}\right)}_{\text{Hermite polynomials orders } l,m} e^{-i(l+m)\phi(z)} \tag{9}$$

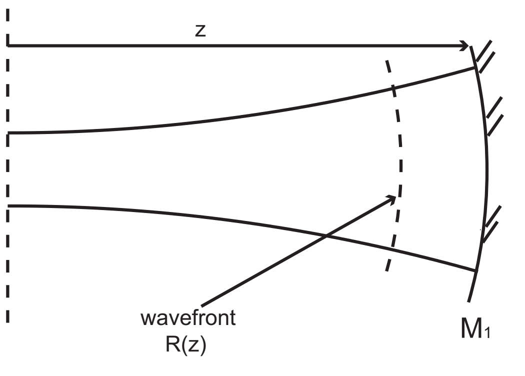
In this case the mode frequencies are offset from integer multiples π by an additional $(l+m+1)\phi(L)c/2\pi L$.



The offset frequency of the mode comb is closely related to the carrier-envelope-offset (CEO) phase.

2 Cavity configurations

The mirror shape should be such that $U(\vec{r}, z)$ satisfies the boundary conditions at the mirrors for all $\vec{r}$. A simple way to think about this is to argue that the mirrors should match the wavefront shape. Then the phase front along which $arg(U(\vec{r}, z)) = 0$ "fits" onto the mirror surface, as illustrated in the figure.



This ensures that the beam will be "reflected back as itself". Clearly a mirror situated to the left of the waist, similarly configured, would provide a "self-reproducing" beam. From this picture, the obvious conditions constraining the cavity length L and mirror radius R_i are:

$$R_i(z_i) = z_i(1 + b^2/z_i^2) \tag{10}$$

Thus:

$$b^2 = (R_1 - z_1)z_1 = (R_2 - z_2)z_2 \tag{11}$$

The cavity length $L = z_1 + z_2$, so:

$$z_1 = \frac{(R_2 - L)L}{R_1 + R_2 - 2L}; \qquad z_2 = \frac{(R_1 - L)L}{R_1 + R_2 - 2L}. \tag{12}$$

Thus finding mirror positions z_1 and z_2 that satisfy these equations and the condition that $b^2 > 0$ will provide a self-reproducing beam that fits into the cavity. Given

$$b^2 = z_1(R_1 - z_1) = \frac{(R_2 - L)L}{R_1 + R_2 - 2L}\left(R_1 - \frac{(R_2 - L)L}{R_1 + R_2 - 2L}\right) > 0 \tag{13}$$

$$\therefore \qquad (1 - L/R_1)(1 - L/R_2)(L/R_1 + L/R_2 - L^2/R_1R_2) > 0 \tag{14}$$

Defining the "cavity g-parameters" $g_i = 1 - L/R_i$; we have

$$g_1g_2(1 - g_1g_2) > 0 \tag{15}$$

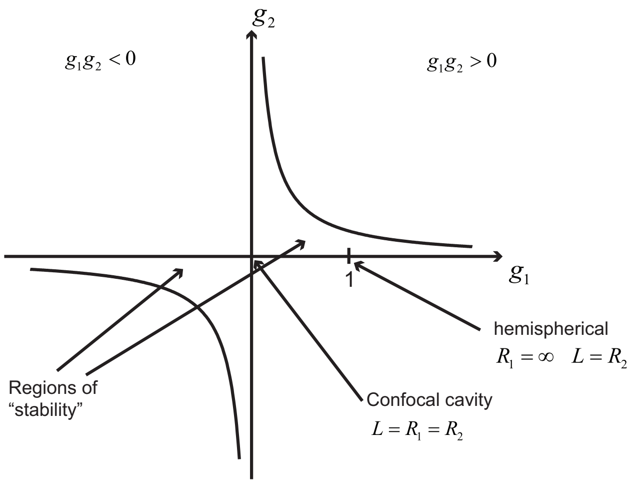
So

$$g_1g_2 > 0 \text{ and } g_1g_2 < 1 \tag{16}$$

or

$$0 < g_1g_2 < 1 \qquad \text{"stability condition" for cavity} \tag{17}$$

This says that, given mirrors of radius R_1 and R_2, we can find a cavity that supports a self reproducing Gaussian beam, provided we choose L to satisfy the above criterias.
The regions of stability are usefully visualized in a graph:



3 General cavities: conditions for low-loss modes

The "self-reproducing" conditions can be made quite general. For an arbitrary cavity that includes whatever elements we choose (lenses, laser rods, Pockels cells, off-axis mirrors etc.), the Gaussian beam should be the same after one round trip.
Let the cavity round trip transfer matrix be $\underline{\underline{T}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then the self-reproducing condition means that the beam q-parameter after a round trip $q^{(+)}$ should be exactly what it was before the round trip. It

is possible to determine the q-parameter after the round trip from that before, in terms of the transfer matrix elements, by means of the relation:

$$q^{(+)} = \frac{Aq + B}{Cq + D} = q. \tag{18}$$

Solving this for q gives:

$$\begin{aligned} Cq^2 + Dq &= Aq + B \\ 2Cq &= (A - D) \pm \sqrt{(A - D)^2 + 4BC}. \end{aligned} \tag{19}$$

Since we know that $AD - BC = 1$ from the unitarity of $\underline{\underline{T}}$ ($\det\underline{\underline{T}} = 1$). this yields

$$q = \frac{A - D}{2C} \pm \frac{1}{2C}\sqrt{(A + D)^2 - 4}. \tag{20}$$

The Gaussian beam <u>must</u> have an imaginary component (though it need not have a real component if $R(z) \to \infty$); it follows that $(A + D)^2 < 4$ is an existence condition for a self-reproducing beam. Since it is obvious that $(A + D)^2 > 0$, the condition may be written as

$$0 < \left(\tfrac{A+D}{2}\right)^2 < 1 \tag{21}$$

$$0 < \tfrac{1}{2}\left|\mathrm{Tr}\underline{\underline{T}}\right| < 1 \qquad \text{Cavity stability criterion} \tag{22}$$

and this holds for any cavity configuration described by $\underline{\underline{T}}$.

4 Mode sizes

The solution to the self reproducing condition gives the beam parameter q. This contains the complete specification of the Gaussian beam, since $q = z - z_{\text{waist}} - ib$. But at what location is this the q-parameter? That depends on how we constructed $\underline{\underline{T}}$. To do this we need to define a <u>reference plane</u> in the cavity, and the solution to the stability condtion gives q at this reference plane. It is clear the $\underline{\underline{T}}$ changes with reference plane choice due to the non-commutativity of the transfer matrices for various elements.
e.g. two mirror cavity:

$$\underline{\underline{T}} = \underline{\underline{T}}_{L_1}\underline{\underline{T}}_{FSP}\underline{\underline{T}}_{L_2}\underline{\underline{T}}_{L_2}\underline{\underline{T}}_{FSP}\underline{\underline{T}}_{L_1} \tag{23}$$

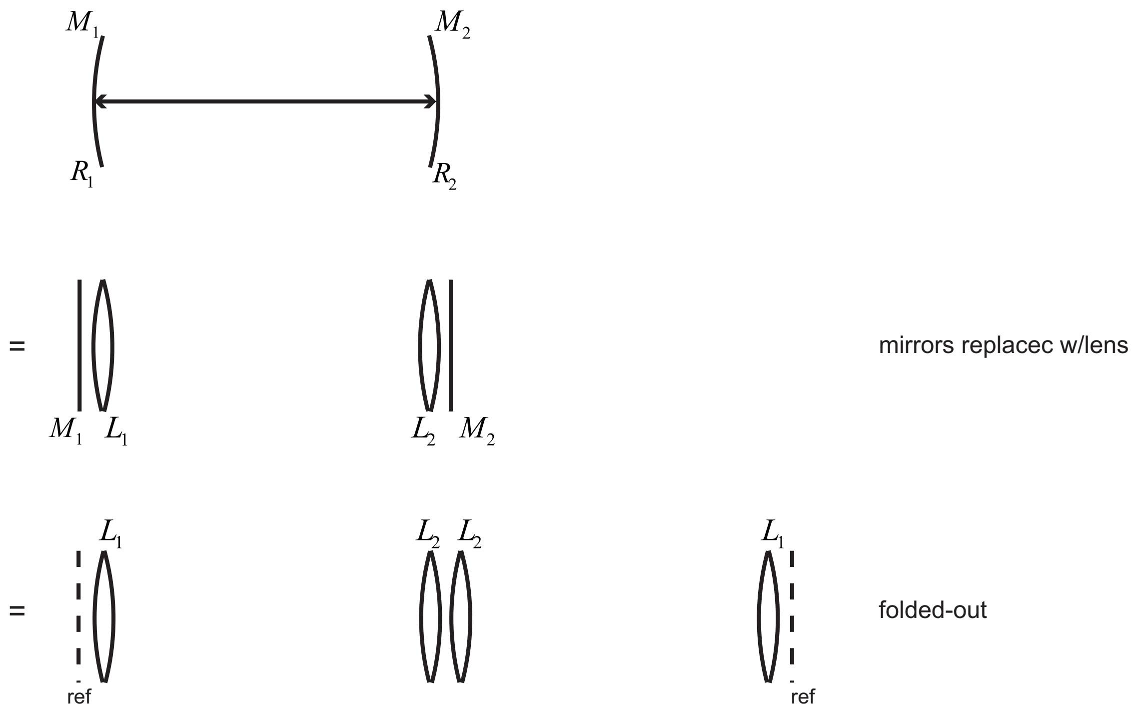
Then this $\underline{\underline{T}}$ will yield the q-parameter at the reference plane indicated just in front of the mirror M_1 (*i.e.* at M_1).

$$q = z - z_{\text{waist}} - ib = \frac{A - D}{2C} - i\frac{\sqrt{4 - (A + D)^2}}{2C}, \tag{24}$$

so

$$z - z_{\text{waist}} = \frac{A - D}{2C} = \frac{1}{R} \tag{25}$$

$$b^2 = \frac{4 - (A + D)^2}{4C} = \frac{\pi w_0^2}{\lambda} \tag{26}$$


M_1
M_2
R_1
R_2
=
M_1 L_1
L_2 M_2
mirrors replacec w/lens
=
L_1
L_2 L_2
L_1
ref
ref
folded-out