Lecture 3: Optics / C2: Quantum Information and Laser Science

November 3, 2008

1 Scalar Diffraction Theory

Consider the wave equation for the electric field:

$$\nabla^2 \vec{E} - \frac{n^2}{c^2}\partial_t^2 \vec{E} = 0 \tag{1}$$

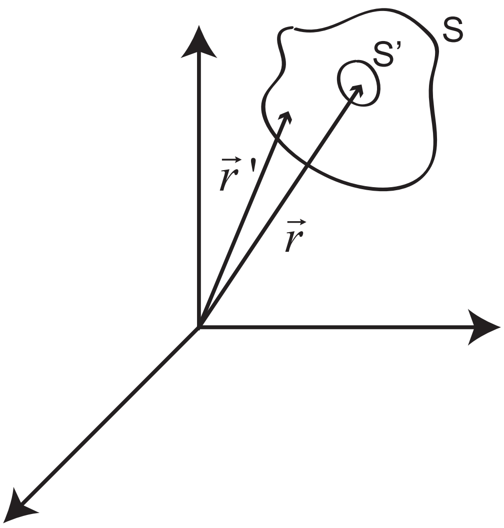
for a plane wave with complex amplitude $\bar{V}(\vec{r}, t)$ and linear polarization $\vec{\epsilon}$. The amplitude must satisfy:

$$\nabla^2 \bar{V} - \frac{n^2}{c^2}\partial_t^2 \bar{V} = 0. \tag{2}$$

Taking a time dependence $\bar{V}(\vec{r}, t) = \bar{V}(\vec{r})e^{-i\omega t}$, then we have the Helmholtz equation:

$$\nabla^2 \bar{V} - \frac{\omega^2}{c^2}\bar{V} = 0. \tag{3}$$

This can be solved by standard techniques using Green's function.
Consider a region of space $\mathcal{V}$ enclosed between two surfaces $\mathcal{S}$ and $\mathcal{S}'$. In this region the Helmholtz equation is solved by two functions $\psi(\vec{r})$ and $\varphi(\vec{r})$.



Now Green's Theorem gives

$$\begin{aligned} \nabla \cdot (\varphi \nabla \psi) &= \varphi \nabla^2 \psi + \nabla \varphi \cdot \nabla \psi && (4) \\ \nabla \cdot (\psi \nabla \varphi) &= \psi \nabla^2 \varphi + \nabla \varphi \cdot \nabla \psi && (5) \end{aligned}$$

Subtracting these

$$
\begin{aligned}
\nabla \cdot (\varphi \nabla \psi - \psi \nabla \varphi) &= \varphi \nabla^2 \psi - \psi \nabla^2 \varphi \\
&= \varphi k^2 \psi - \psi k^2 \varphi \qquad (k^2 = \omega^2/c^2) \\
&= 0 \qquad (6)
\end{aligned}
$$

So that

$$\int_{\mathcal{V}} d^3x \, \nabla \cdot (\varphi \nabla \psi - \psi \nabla \varphi) = 0. \qquad (7)$$

Using the divergence theorem:

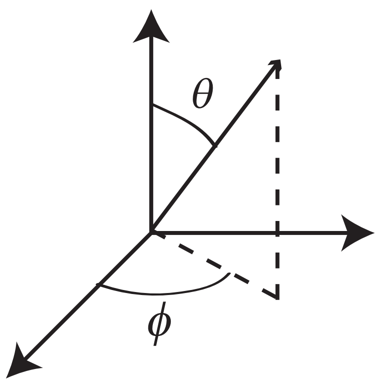
$$\int_{\mathcal{V}} d^3x \, \nabla \cdot (\varphi \nabla \psi - \psi \nabla \varphi) = \int_{\mathcal{S}+\mathcal{S}'} d\vec{s} \cdot (\varphi \nabla \psi - \psi \nabla \varphi) = 0. \qquad (8)$$

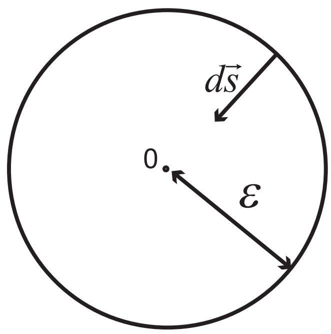
where $d\vec{s}$ is a surface normal as the differential element. Thus

$$\int_{\mathcal{S}} d\vec{s} \cdot (\varphi \nabla \psi - \psi \nabla \varphi) = -\int_{\mathcal{S}'} d\vec{s} \cdot (\varphi \nabla \psi - \psi \nabla \varphi) \qquad (9)$$

Let $\mathcal{S}'$ be a sphere with radius ϵ, centered at 0, with unit radial vector $\hat{r}$. The differential surface normal is then

$$d\vec{s} = \hat{r} \epsilon^2 \, d\phi d\theta \, \sin\theta. \qquad (10)$$





Now one solution for the field in a spherical cavity is just a spherical wave. Therefore, let $\psi = e^{ik\epsilon}/\epsilon$, and $\hat{r}$ be the unit radial vector from 0, so that

$$\hat{r} \cdot \nabla \psi = \frac{\partial \psi}{\partial \epsilon} = \left(ik - \frac{1}{\epsilon} \right) \frac{e^{ik\epsilon}}{\epsilon}. \qquad (11)$$

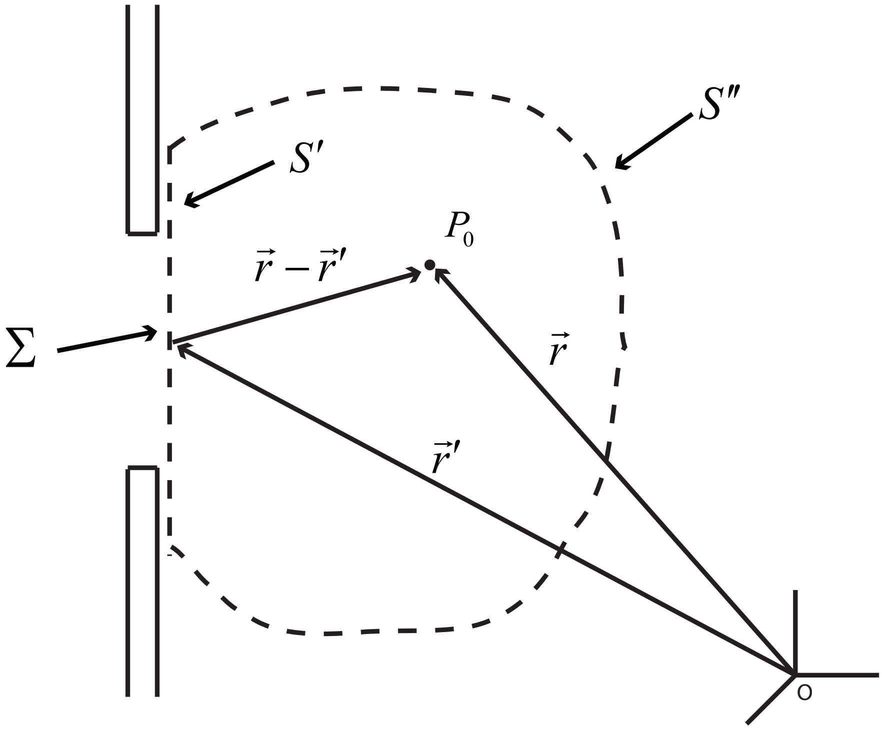
Thus

$$
\begin{aligned}
\int_{\mathcal{S}'} d\vec{s} \cdot (\varphi \nabla \psi - \psi \nabla \varphi) &= -\int_0^{2\pi} d\phi \int_0^{pi} d\theta \, \sin\theta \epsilon^2 \left[\varphi(\vec{r}) \left(ik - \frac{1}{\epsilon} \right) \frac{e^{ik\epsilon}}{\epsilon} - \frac{e^{ik\epsilon}}{\epsilon} \frac{\partial \varphi(\vec{r})}{\partial \epsilon} \right] \\
&= -2\pi 2 \left\{ \varphi(\vec{r}) ik\epsilon^2 - \varphi(\vec{r}) \right\} e^{ik\epsilon} + 4\pi\epsilon \frac{\partial \varphi(\vec{r})}{\partial \epsilon} \qquad (12) \\
\lim_{\epsilon \to 0} \int_{\mathcal{S}'} d\vec{s} \cdot (\varphi \nabla \psi - \psi \nabla \varphi) &= -4\pi \varphi(\vec{r}) \qquad (13)
\end{aligned}
$$

So that

$$\varphi(\vec{r}) = \frac{1}{4\pi} \int_{\mathcal{S}} d\vec{s} \cdot \varphi(\vec{r}') \nabla \left\{ \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \right\} - \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \nabla \varphi(\vec{r}') \qquad (14)$$

This is the **Helmholtz – Kirchhoff Theorem**.



We now apply this to the case of diffraction from an aperture. The goal is to find the field at P_0 (location $\vec{r}$ due to the field in the aperture Σ (parts in the aperture are at location $\vec{r}'$). The aperture lies on the surface of a volume $\mathcal{V}$ that is bounded by $\Sigma + \mathcal{S}' + \mathcal{S}''$. It is taken that the field is prescribed on Σ; *i.e.*, $\varphi(\vec{r}')$ and $\frac{\partial\varphi}{\partial n}(\vec{r}')$ are known, and that both are identically zero in the region $\mathcal{S}'$ of the opaque screen. [1] Then

$$\varphi(\vec{r}) = \int_\Sigma d\vec{s} \cdot \varphi(\vec{r}')\nabla\left\{\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\right\} - \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\nabla\varphi(\vec{r}') \tag{15}$$

Taking the limit as $\mathcal{S}'' \to$ a very large distance away from the aperture screen, there is no contribution from fields at this surface. Then it follows

$$\varphi(\vec{r}) = \int_{\text{aperture}} dxdy\ \varphi(\vec{r}')\frac{\partial}{\partial z}\left\{\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\right\} - \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\frac{\partial}{\partial z}\varphi(\vec{r}') \tag{16}$$

Then using

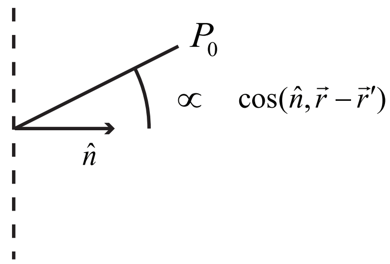
$$\frac{\partial}{\partial z}\left\{\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\right\} = \left\{ik - \frac{1}{|\vec{r}-\vec{r}'|}\right\}\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\cos(\hat{n}, \vec{r}-\vec{r}') \tag{17}$$

where $\hat{n}$ is a unit vector in the direction of the surface normal from Σ. The right-hand-side is usually approximated, for distances $|\vec{r}-\vec{r}'|$ that are much greater than on wavelength $(2\pi/k)$ away from the surface Σ, as

$$\frac{\partial}{\partial z}\left\{\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\right\} \approx ik\cos(\hat{n}, \vec{r}-\vec{r}')\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \tag{18}$$

Evaluating the second term in the Helmholtz-Kirchhoff term requires some approximation. Considering the case that the derivative is taken in the normal to the aperture, and that this lies in the x-y plane, then we can approximate this as a derivative with respect to the z-axis. In that case, the dominant contribution to the derivative will likely come from the rapid change in the field across a wavelength.

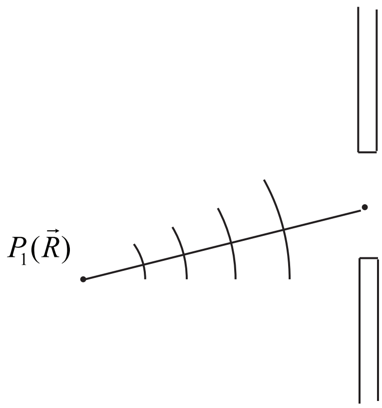
[1] Unfortunately, this overspecifies the problem. The mixed boundary conditions lead to the situation that if both $\varphi(\vec{r}')$ and $\frac{\partial\varphi}{\partial n}(\vec{r}')$ are zero on some part of the surface, then $\varphi(\vec{r}')$ will vanish everywhere. There are more sophisticated ways around this so-called "saltus" or "jump" problem than we shall use here.



Therefore, it is reasonable to approximate the term by

$$\nabla\varphi(\vec{r}') \cong ik\varphi(\vec{r}')\cos(\hat{n}, \vec{r}' - \vec{R}), \tag{19}$$

Provided the field "envelope" is slowly varying on the scale of a wavelength.



To develop this, first consider the case of a point source at $\vec{R}$ on the left half space of Σ, from which a spherical wave diverges . In this case,

$$\varphi(\vec{r}') = \frac{e^{ik|\vec{r}' - \vec{R}|}}{|\vec{r}' - \vec{R}|}, \tag{20}$$

so that

$$\frac{\partial}{\partial z}\left\{\frac{e^{ik|\vec{r} - \vec{R}|}}{|\vec{r} - \vec{R}|}\right\} \approx ik\cos(\hat{n}, \vec{r} - \vec{R})\frac{e^{ik|\vec{r} - \vec{R}|}}{|\vec{r} - \vec{R}|}. \tag{21}$$

For the case of an extended source, we may approximate the derivative in the aperture by replacing the spherical wave by the actual field,

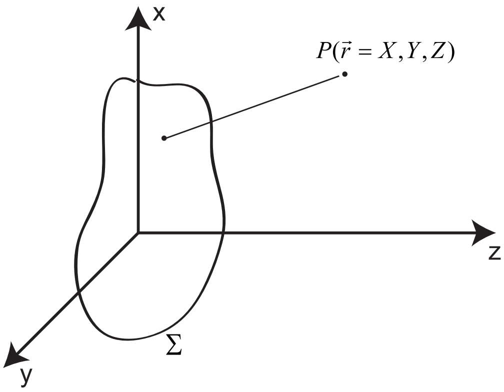
$$\frac{\partial}{\partial z}\phi(\vec{r}) \approx ik\cos(\hat{n}, \vec{r} - \vec{R})\phi(\vec{r}). \tag{22}$$

With this "slowly varying envelope" approximation, the diffraction formula simplifies to :

$$\varphi(\vec{r}) = \int_\Sigma dxdy\ \varphi(\vec{r}')\frac{e^{ik|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|}ik\ \underbrace{\left\{\cos(\hat{n}, \vec{r} - \vec{r}') - \cos(\hat{n}, \vec{r}' - \vec{R})\right\}}_{\approx 2 \text{ for small angles from the surface normal}} \tag{23}$$

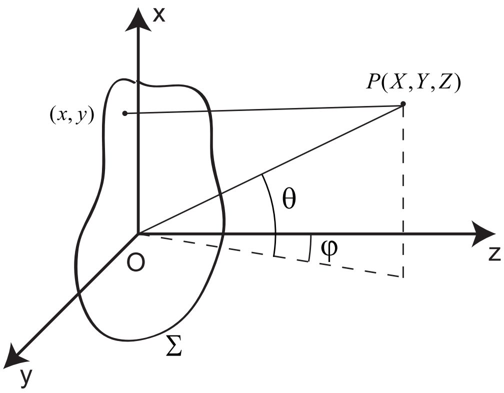
If we stay in the paraxial regime, the obliquity factors (the cosines) can be set equal to unity (and -1). We end up with the *Kirchoff diffraction integral*

$$\varphi(\vec{r}) = \int_\Sigma dxdy\ \varphi(x, y)\frac{e^{ik\sqrt{(X-x)^2+(Y-y)^2+Z^2}}}{Z} \tag{24}$$



2 Fraunhofer Diffraction

The Kirchoff diffraction integral can be approximated to a familiar form under a certain set of conditions, called the Fraunhofer conditions. (x, y) in aperture Σ, $P(X, Y, Z)$ at observation part, θ =angle between



the z-axis and OP.
Then the argument in the exponent in the integral is :

$$\left((X-x)^2+(Y-y)^2+Z^2\right)^{1/2} = \left[(X^2+Y^2+Z^2)-2Xx-2Yy+(x^2+y^2)\right]^{1/2} \tag{25}$$

Let the $X^2 + Y^2 + Z^2 = R^2$ = distance between O and P

$$\left((X-x)^2+(Y-y)^2+Z^2\right)^{1/2} = R\left(1-\frac{2Xx}{R^2}-\frac{2Yy}{R^2}+\frac{x^2+y^2}{R^2}\right)^{1/2} \tag{26}$$

For $R \gg$maximum radius of the aperture, a ...... distance of P from the z-axis:

$$\left((X-x)^2+(Y-y)^2+Z^2\right)^{1/2} \cong R-\frac{Xx}{R}-\frac{Yy}{R}+\frac{x^2+y^2}{2R} \tag{27}$$

It is often (though not always) convenient to write $X/R = \sin\theta$, $Y/R = \sin\varphi$, with the angles as indicated on the diagram. Thus the Kirchoff integral becomes:

$$\varphi(X,Y,Z) \cong \frac{2ik}{4\pi Z}\int_\Sigma dxdy\ \varphi(x,y,0)e^{ikR}\underbrace{e^{-ik\frac{X}{R}x}e^{-ik\frac{Y}{R}y}}_{\text{Fourier Transform-like terms}}e^{ik\frac{x^2+y^2}{2R}} \tag{28}$$

In the case of a planar 1-D geometry, this simplifies further; since we only need consider points which have $y = 0$, $Y = 0$. Then:

$$\varphi(X,Z) \cong \frac{1}{\lambda Z}\int_\Sigma dx\ \varphi(x,0)e^{ikR}e^{-ik\sin\theta x}e^{ik\frac{x^2}{2R}} \tag{29}$$

The integral looks exactly like a Fourier transform except for the term $\exp(ikx^2/2R)$.
This can be eliminated in several ways:

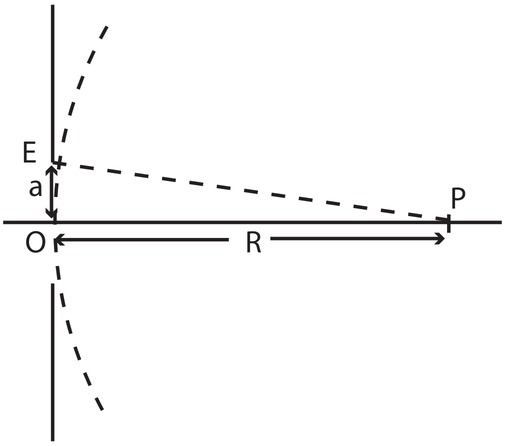
1. It can be ignored – if it represents a sufficiently small phase accumulation over the range of integration.

   Let us say the maximum extent of the aperture is a. Then the maximum phase due to this term is:

$$\frac{\pi a^2}{\lambda R} = \mathcal{F} \tag{30}$$

   where $\mathcal{F}$ is called the *Fresnel number* of the arrangement.

   It gives the number of half wavelengths that the path from the edge of the aperture to the enter of the observation plane differs from the path from the center of the aperture to the center of the observation plane.



$$\begin{aligned} EP - OP &= \sqrt{R^2 + a^2} - R \\ &\approx R + \frac{a^2}{2R} - R \\ &\approx \frac{a^2}{2R} \\ \therefore \quad \frac{EP - OP}{\lambda/2} &= \mathcal{F} \end{aligned}$$

   Ideally $\mathcal{F} = 0$, but it is usually sufficient to allow $\mathcal{F} \ll 1$. This is called the *Far Zone* approximation.

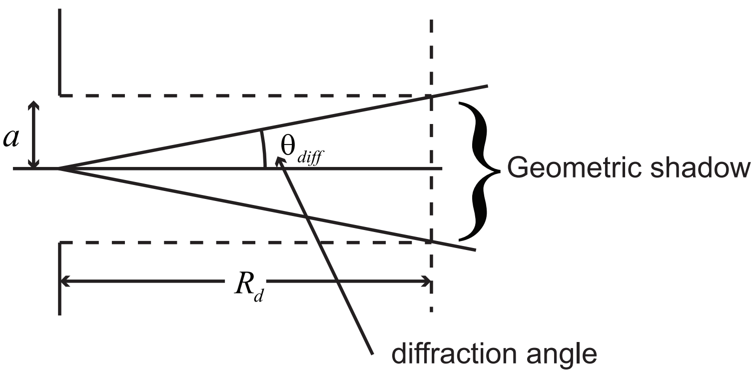
   It represents a condition on the distance we must place the observation place from the aperture in order to see Fourier-transform-like diffraction patterns.

   The distance for which the Far-Zone approximation holds is called the *Rayleigh distance*, and is the distance at which the diffracted beam becomes larger than the geometrical shadow of the aperture.

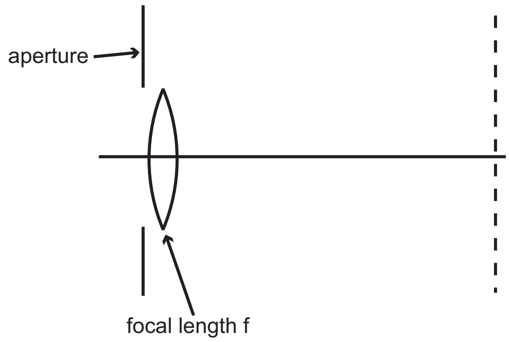
$$\begin{aligned} \theta_{geo} &\approx a/R \\ \theta_{diff} &= \lambda/a \\ \frac{\theta_{geo}}{\theta_{diff}} &= \frac{a}{R}\frac{a}{\lambda} = \mathcal{F} \end{aligned}$$

   The Rayleigh distance R_d is the radius of R that gives $\mathcal{F} = 1$.

$$R_d = \frac{a^2}{\lambda} \tag{31}$$



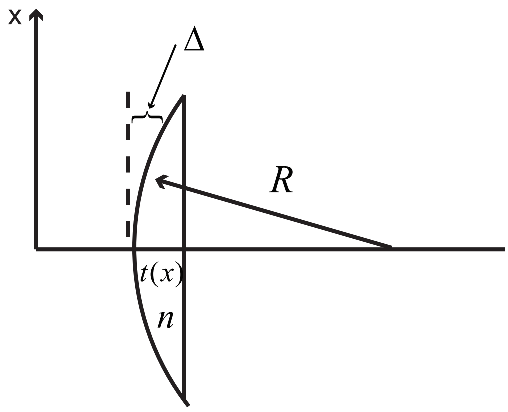
2. The second way to eliminate the quadratic phase term in the integral is to put a lens immediately after the aperture:



The field at the observation plane needs to be evaluated again in this case. We note that the field at the aperture $\varphi(x, y)$ is multiplied by the transfer function of the lens $t_l(x, y)$, and the beam allowed to propagate to the observation plane. The transfer function of the lens is

$$t_l(x, y) = e^{-ik(x^2+y^2)/2f} \tag{32}$$

To see that this form is reasonable consider a special case where the lens is plano-convex, of material with refractive index n , with one surface radius R.



The phase $\xi(x)$ at position x accumulated by a plane wave incident from the left of this lens is

$\xi(x) = k\Delta(x) + nkt(x)$

$$\begin{aligned} \xi(x) &= \underbrace{kt(x)n}_{\text{path in lens}} + \underbrace{k\left(t(0) - t(x)\right)}_{\text{path to lens surface}} \\ \xi(x) &= k(n-1)t(x) + kt(0) \end{aligned} \tag{33}$$

and

$$\begin{aligned} t(x) &= t(0) - \left(R - \sqrt{R^2 - x^2}\right) \\ &= t(0) - R + R\left(1 + \frac{x^2}{R^2}\right)^{1/2} \\ &\cong t(0) - R + R + \frac{x^2}{2R} \\ \therefore \quad \xi(x) &= k(n-1)t(0) + kt(0) - k(n-1)\frac{x^2}{2R} \\ \xi(x) &= knt(0) - k(n-1)\frac{x^2}{2R} \end{aligned} \tag{34}$$

If we now let $(n-1)/R = 1/f$, where f is the focal length of the lens (see 1st year optics course):

$$t_l(x, y) = e^{iknt(0)}e^{-ik(n-1)x^2/2R} = e^{i\varphi_0}e^{-ikx^2/2f} \tag{35}$$

Inserting the lens transmission function into the integral of eq. 29 gives (keeping to 1-D still)

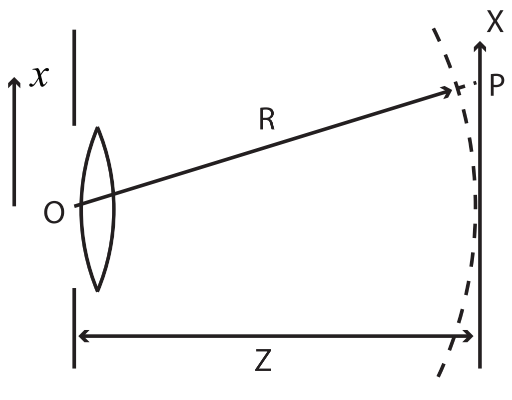
$$\varphi(X, Z) = \frac{e^{ikR}}{\lambda Z}\int_\Sigma dx\ \varphi(x, 0)e^{i\varphi_0}e^{-ikx^2/2f}e^{-ik\sin\theta x}e^{ik\frac{x^2}{2R}} \tag{36}$$

It is clear that if $f = R$ all quadratic terms in x in the integral phases disappear, and we are left with an F.T.

$$\varphi(X, Z) = \frac{e^{ikR + i\varphi_0}}{\lambda Z}\int_\Sigma dx\ \varphi(x, 0)e^{-ik\sin\theta x} \tag{37}$$

Note that this is not a "true" Fourier transform of $\varphi(x, 0)$, because of the additional phase term outside the integral $\exp(ikR + i\varphi_0)$. This gives a field-angle(X)-dependent phase across the observation plane.

$$\begin{aligned} R &= \sqrt{Z^2 + X^2} \\ &\approx Z + \frac{X^2}{2Z} \end{aligned}$$



Also the factor $1/Z$ accounts for the 'spreading' of energy across the observation plane. To see this we need to go back to the full 2-D model:

$$\begin{aligned}\varphi(X,Y,Z) &= \frac{e^{ikR}}{\lambda Z}\int_\Sigma dxdy\ \varphi(x,y)e^{i\varphi_0}e^{i\varphi_0}e^{-ik\sin\theta x-ik\sin\varphi y} \\ &= \frac{e^{ikR}}{\lambda Z}\tilde{\varphi}\left(\frac{X}{\lambda Z},\frac{Y}{\lambda Z}\right) \end{aligned} \quad (38)$$

recalling that $\sin\theta = X/R \cong X/Z$ and $\sin\varphi = Y/R \cong Y/Z$ in the paraxial limit.

Then it must be the case that the total energy transmitted through the aperture is the same as the total energy recorded in the observation plane.

$$\begin{aligned}\int dxdy\ |\varphi(x,y,0)|^2 &= \int dXdY\ |\varphi(X,Y,Z)|^2 \\ &= \int dXdY\ \frac{1}{\lambda^2 Z^2}\left|\tilde{\varphi}\left(\frac{X}{\lambda Z},\frac{Y}{\lambda Z}\right)\right|^2 \end{aligned} \quad (39)$$

Let $\bar{X} = X/\lambda Z$, $\bar{Y} = Y/\lambda Z$, then

$$\int dxdy\ |\varphi(x,y,0)|^2 = \int d\bar{X}d\bar{Y}|\tilde{\varphi}(\bar{X},\bar{Y})|^2 \quad (40)$$

which is true by Parseval's theorem.

3. The last configuration I wish to consider is one relevant to imaging and illumination - not just in the optical regime. It is the sort of geometry that arises in most situations where laser light is used to probe an object. Examples include confocal and nonlinear microscopy, optical tweezers and laser plasma excitation, as well as the preparation of, say, photonic qubits coded in angular momentum. The physics is closely related to that of the previous example, in that the light exiting the aperture is nearly a spherical wave. Here, though, this is achieved by illuminating the aperture with a converging spherical wave, rather than placing a lens after it. The aperture may then be modelled by a transmission function, T, that specifies how the field at each point in the aperture is modified. It is immediately evident how this is related to imaging instruments. Because of this method of illumination, the field in the observation plane can be made a Fourier Transform of the field in the object plane, which encodes the structure of the transmission function.

   Denote the field in the aperture by

$$\varphi(x,y,0) = \psi_{inc}(x,y,0)T(x,y) \quad (41)$$

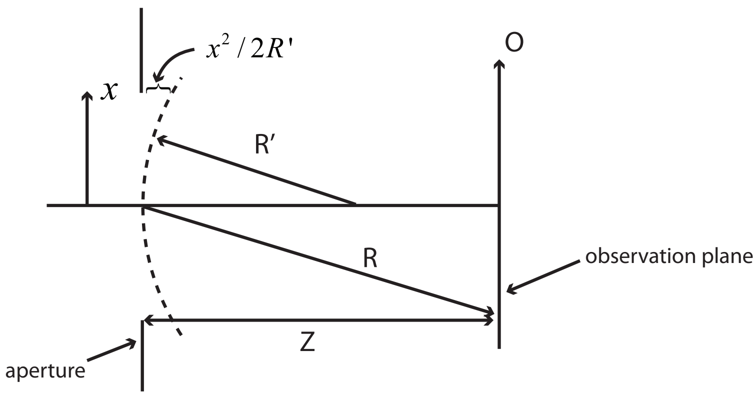
   where $\psi_{inc}(x,y,0)$ is the incident spherical wave and $T(x,y)$ is the aperture transmission function.

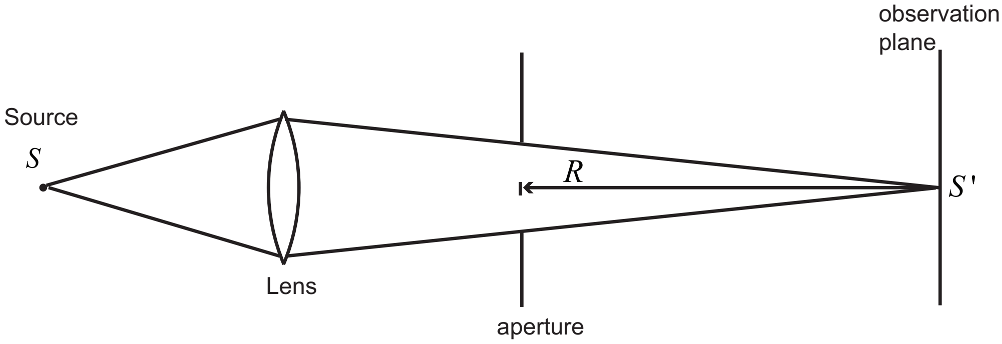
   In the paraxial approximation

$$\psi_{inc}(x,y,0) = e^{-ikR'}\frac{e^{-ik(x^2+y^2)/2R'}}{\lambda R'} \quad (42)$$

   So it is clear that if $R' = R$, then the field at the observation plane is:

$$\begin{aligned}\varphi(X,Y,Z) &= \frac{e^{ikR}}{\lambda Z}\int_\Sigma dxdy\ T(x,y)e^{-ik(x^2+y^2)/2R'}\frac{e^{-ikR'}}{\lambda R'}e^{-ik\sin\theta x-ik\sin\varphi y}e^{ik(x^2+y^2)/2R} \\ R' \cong Z \qquad &= \frac{e^{ik(R-R')}}{\lambda^2 Z^2}\int_\Sigma dxdy\ T(x,y)e^{-ik\sin\theta x-ik\sin\varphi y}e^{ik\frac{x^2+y^2}{2}\left(\frac{1}{R}-\frac{1}{R'}\right)} \\ &= \frac{1}{\lambda^2 Z^2}\int_\Sigma dxdy\ T(x,y)e^{-ik\sin\theta x-ik\sin\varphi y} \end{aligned} \quad (43)$$



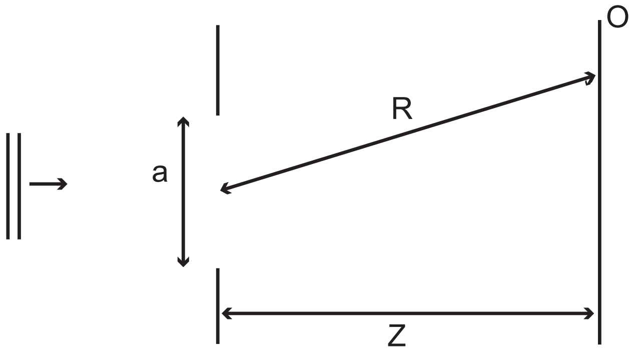


This time the Fourier Transform is exact, and there are no extraneous field-angle dependent phase terms. But how do we set up this configuration? Simple, we need a converging beam on the aperture, so we use a lens.

If S' is arranged to be an image of S, then the above condition is satisfied.

Thus the 3 configurations for obtaining Fourier Transform relations between the aperture function and the field in the observation plane - the conditions of *Fraunhofer diffraction* - are:

1. Far-zone - plane wave illumination

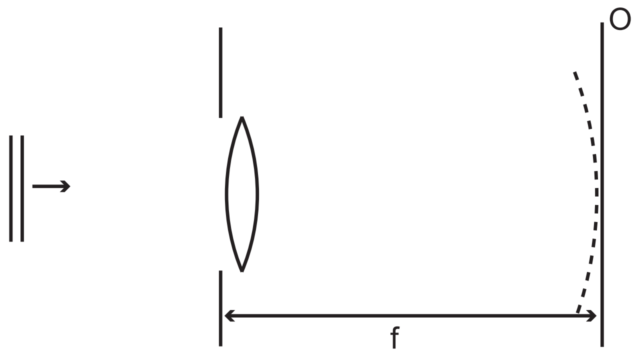


This is defined by the approximation that the Fresnel number be very small:

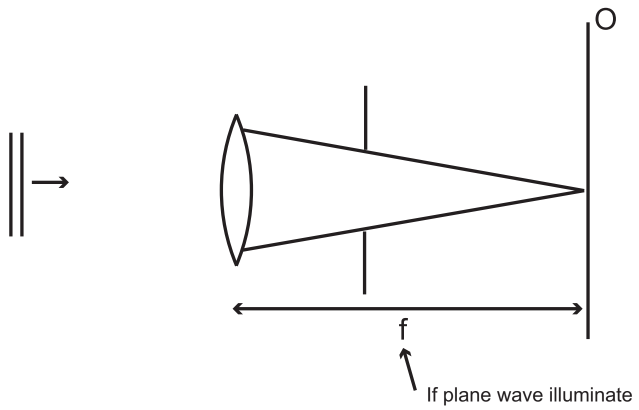
$$\mathcal{F} = \frac{a^2}{\lambda R} \ll 1. \tag{44}$$

2. Lens at aperture - plane wave illumination

   Note that the field at the observation plane is not the Fourier-Transform of the field at the aperture because of field-angle dependent phase. The modulus of the field, however, is the modulus of this Fourier Transform.

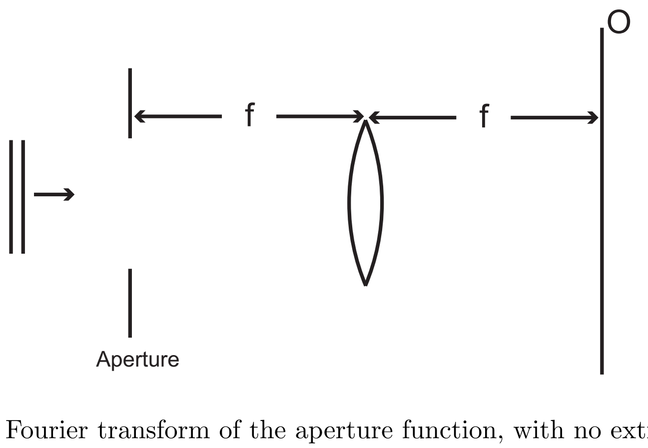


3. Lens before aperture - plane or diverging illumination at the lens



The observation plane is located at the source image. The field at this plane is an exact Fourier Transform of the field at the aperture.

The most useful case, and the most general, is when the lens is place in front of the aperture by its focal length, and the observation plane is in the back focal plane of the lens.
In this case



The field at O is an exact Fourier transform of the aperture function, with no extraneous phases, provided the aperture is illuminated with plane waves. This example will be explored in the homework