Optics Lecture 1

October 7, 2008

1 Review of Electrodynamics

The basic of optics is the wave motion of electromagnetic fields. We therefore begin by reviewing the origin and nature of electromagnetic radiation:

Maxwell's Equations:

$$\nabla \cdot \vec{D} = \rho_f \quad (1)$$
$$\nabla \cdot \vec{H} = 0 \quad (2)$$
$$\nabla \times \vec{E} = -\partial_t \vec{B} \quad (3)$$
$$\nabla \times \vec{H} = \vec{v}_v + \partial_t \vec{D} \quad (4)$$

These are supplemented by the constitutive relations:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad (5)$$
$$\vec{H} = \vec{B} - \vec{M} \quad (6)$$

where ϵ, μ are the dielectric permittivity and magnetic permeability respectively if

$$\vec{M} = \chi_m \vec{H}; \qquad \mu = \mu_r \mu_0 \qquad \mu_r = \text{relative permeability} \quad (7)$$
$$\vec{P} = \chi_e \vec{E}; \qquad \epsilon = \epsilon_r \epsilon_0 \qquad \epsilon_r = \text{relative permittivity} \quad (8)$$

The bound charges and currents are specified by the polarization density $\vec{P}$ and the magnetic dipole moment density $\vec{M}$.

These equations lead to the wave equations for the electric and magnetic fields:

$$\nabla \times \nabla \times \vec{E} = -\partial_t \nabla \times \vec{b} = -\partial_t \nabla \times \mu \vec{H} = -\mu \partial_t^2 \vec{D} = -\mu \epsilon \partial_t^2 \vec{E} \quad (9)$$

in a region with no free charges, and linear response media - a linear dielectric.

But:

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \quad (10)$$

since $\nabla \cdot \vec{E} = 0$ in dielectrics, so

$$-\nabla^2 \vec{E} = -\mu \epsilon \partial_t^2 \vec{E} \quad (11)$$

$$\nabla^2 \vec{E} = \frac{1}{v^2} \partial_t^2 \vec{E} \qquad v = \frac{1}{\sqrt{\mu\epsilon}} \text{ is phase velocity} \quad (12)$$

Similarly

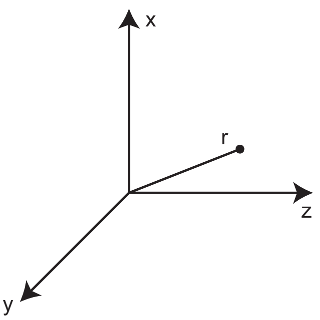
$$\nabla^2 \vec{H} = \frac{1}{v^2} \partial_t^2 \vec{H} \quad (13)$$

The refractive index is

$$n = \frac{c}{v} = \sqrt{\mu_r \epsilon_r} = \sqrt{\epsilon_r} \quad (14)$$

These equations can be solved by plane waves:

$$\vec{E} = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t) \quad (15)$$



where $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, and $\hat{x}$, $\hat{y}$, $\hat{z}$ are unit vectors in the Cartesian frame.

Thus from the wave equations:

$$\left(\partial_x^2 + \partial_y^2 + \partial_z^2 - \frac{n^2}{c^2}\partial_t^2\right)\vec{E}(\vec{r},t) = 0 \tag{16}$$

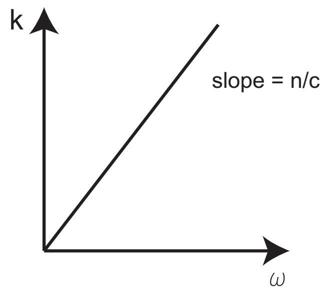
we get

$$\left(k_x^2 + k_y^2 + k_z^2 - \frac{n^2\omega^2}{c^2}\right)\vec{E}(\vec{r},t) = 0 \tag{17}$$

If this is true for all x, y, z, t, then the dispersion relation must hold:

$$k^2 = \frac{n^2\omega^2}{c^2} \tag{18}$$

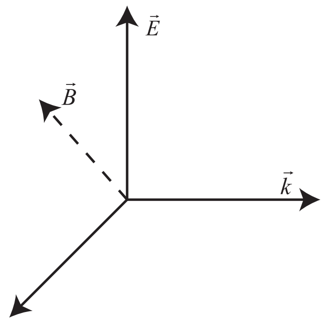
where k = wavenumber, ω = (angular) frequency.



Using Maxiwell equation (1), we find the transversality condition

$$\nabla \cdot \vec{E} = \vec{k} \cdot \vec{E} = 0 \tag{19}$$

So $\vec{k} \perp \vec{E}$, and it is always true that $\nabla \cdot \vec{B} = 0$, so $\vec{k} \perp \vec{B}$ also. Further from the last two Maxwell equations



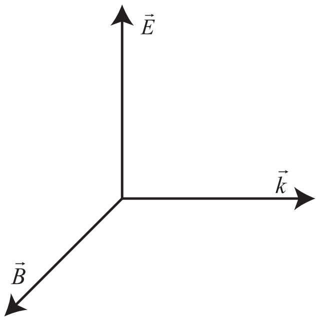
(3) and (4), we have

$$\nabla \times \vec{E} = i\vec{k} \times \vec{E} = -\partial_t \vec{B} = i\omega\vec{B} \tag{20}$$

and

$$\nabla \times \vec{H} = i\vec{k} \times \vec{H} = \partial_t \vec{D} = i\omega\epsilon\vec{E} \tag{21}$$

Eqn (20) says that $\vec{B}$ is $\perp$ to the plane containing $\vec{k}$ and $\vec{E}$, so $\vec{B} \perp \vec{E}$. Thus $\vec{k}$, $\vec{E}$, $\vec{B}$ are all mutually perpendicular.



$$\frac{|\vec{E}|}{|\vec{H}|} = \frac{\mu\omega}{k} = \sqrt{\frac{\mu}{\epsilon}} = Z_0 \tag{22}$$

$$Z_0 = \text{ impedance } = 377\Omega \text{ in free space}$$

Now $\vec{k}$ is the direction of propagation of the field.

Proof: At time t, the phase at position $\vec{r}$ is, say $\phi =$ constant. At time t', the phase at postion $\vec{r'}$ is ϕ'.

Letting $\vec{r'} = \vec{r} + \delta\vec{r}$; $t' = t + \delta t$; so $\omega = \vec{k} \cdot \delta\vec{r}/\delta t$. But in the limit $\delta t \to 0$, $\delta\vec{r}/\delta t = d\vec{r}/dt =$ velocity fo phase front $= \vec{v}$

$\therefore \quad \omega = |\vec{k}||\vec{v}|\cos(\vec{k}, \vec{v})$

But also: $k = n\omega/c = |\vec{k}|$

$\therefore \quad \cos(\vec{k}, \vec{v}) = 1$; so $\vec{v}$ and $\vec{k}$ are in the same direction.

2 Poynting Vector

We have that $\vec{E} \times \vec{H}$ points in the direction of $\vec{k}$, the direction of propagation of the plane wave (in the linear, isotropic, homogeneous medium we have assumed). This vector is called the Poynting vector.

$$\vec{S} = \vec{E} \times \vec{H}. \tag{23}$$

It measures the flow of energy in the electromagnetic field.

Recall that the energy density in a field in a dielectric (in a non-dispersive medium) is:

$$v = \frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2. \tag{24}$$

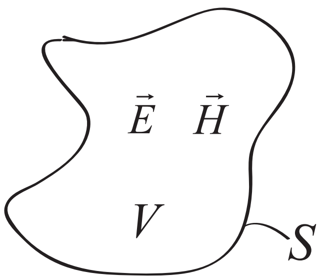
Now from Maxwell (3) and (4):

$$\vec{E} \cdot (\nabla \times \vec{H}) - \vec{H} \cdot (\nabla \times \vec{E}) = \vec{E} \cdot \partial_t \vec{D} + \vec{H} \cdot \partial_t \vec{B} \tag{25}$$

$$\begin{aligned} -\nabla \cdot (\vec{E} \times \vec{H}) &= \vec{E} \cdot \partial_t(\epsilon\vec{E} + \vec{H} \cdot \partial_t(\mu\vec{H}) \\ &= \frac{1}{2}\epsilon\partial_t E^2 + \frac{1}{2}\mu\partial_t H^2 \\ &= \partial_t v \end{aligned} \tag{26}$$

Integrate over an arbitrary volumn $\mathcal{V}$ with surface $\mathcal{S}$

$$-\int_{\mathcal{V}} d^3x \, \nabla \cdot (\vec{E} \times \vec{H}) = \partial_t \int_{\mathcal{V}} d^3x \, v \tag{27}$$



Using the divergence theorem:

$$-\int_{\mathcal{S}} d\vec{x} \cdot (\vec{E} \times \vec{H}) = \partial_t \int_{\mathcal{V}} d^3x\, v \qquad \text{(Poynting's Theorem)} \tag{28}$$

The interpretation of this equation is:

Right-hand-side time rate of change of electromagnetic energy inside the volumn $\mathcal{V}$

Left-hand-side energy leaving the volumn $\mathcal{V}$ through the surface $\mathcal{S}$ per unit time in the electromagnetic field

Therefore $\vec{S} = \vec{E} \times \vec{H}$ is the energy per unit time per unit area across $\mathcal{S}$.

Typically we are interested in time-averaged qualities. This is because the EM field oscillates very rapidly compared with the detectors and experiments we are able to access. Thus we define a radiant intensity:

$$\vec{I} = \langle \vec{S} \rangle = \frac{1}{2T} \int_{-T}^{T} dt \vec{S}(t) \tag{29}$$

where $2T \gg 2\pi/\omega$.

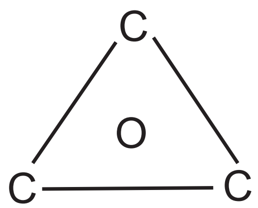
For the case we have just described (with cosinusoidaly varying fields):

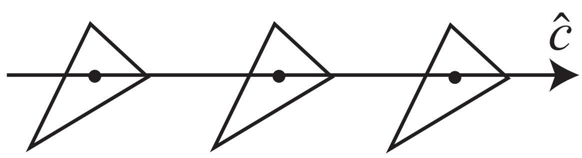
$$\begin{aligned} \vec{I} = I\vec{k} &= \frac{1}{2T} \int_{-T}^{T} E_0 H_0 \cos^2(\vec{k} \cdot \vec{r} - \omega t)\vec{k} \\ &= \frac{1}{2} E_0 H_0 \vec{k} \\ \vec{I} &= \frac{1}{2} \frac{n}{Z_0} E_0^2 \vec{k} \qquad (\text{Watts m}^{-2}) \\ \vec{I} &= \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} E_0^2 \vec{k} \end{aligned} \tag{30}$$

3 Non-isotropic media

There exists an important classes of dielectrics for which the refractive index is a function of the direction of propagation of the plane wave. These are called bi-refringent media, and are used in polarizing optics.

These materials are often crystals, and it is easy to see why they might exhibit birefringence: For example in calcite ($CaCO_3$) the arrangement of the atoms is such that the carbonate group forms a plane. Calcite



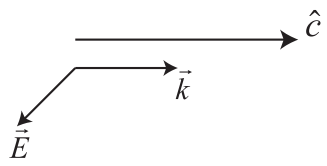


consists of a parallel arrangement of such groups.

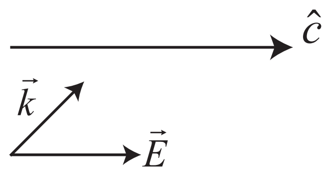
The major axis of symmetry - the optic axis - is parallel to the normal to these planes.

The refractive index of the medium is a measure of the ability of a field to create dipoles in the medium, and it is clear that for calcite the effect of an applied field will be larger when it is directed perpendicular to the optic axis rather than along it.

Case 1. $\vec{E}\perp\hat{c}$



Case 2. $\vec{E}\|\hat{c}$



4 Plane wave solutions in non-isotropic media

Let

$$\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t) \tag{31}$$

$$\vec{D} = \vec{D}_0 \cos(\vec{k}\cdot\vec{r} - \omega t) \tag{32}$$

$$\text{where} \quad \vec{k} = \frac{n\omega}{c}\hat{s} \qquad (\hat{s} = \text{unit vector in direction of } \vec{k})$$

Then using:

$$\nabla\times\nabla\times\vec{E} = \mu\partial_t^2\vec{D} \tag{33}$$

$$\vec{k}\times\vec{k}\times\vec{E} = \mu\omega^2\vec{D} \tag{34}$$

$$\frac{n^2\omega^2}{c^2}\hat{s}\times\hat{s}\times\vec{E} = \mu\omega^2\vec{D} \tag{35}$$

Now a vector identity yields $\hat{s}\times\hat{s}\times\vec{E} = \hat{s}(\hat{s}\cdot\vec{E}) - \vec{E}(\hat{s}\cdot\hat{s})$

$$\therefore \qquad \vec{D} = n^2\epsilon_0(\vec{E} - \hat{s}(\hat{s}\cdot\vec{E})) \tag{36}$$

The important result here is to note that $\hat{s}\cdot\vec{E}\neq 0$; since we have not assumed that $\nabla\epsilon = 0$.

Moreover this formula shows that $\vec{D}$ and $\vec{E}$ are not parallel, in general; since the applied field induces a dipole moment in a slightly different direction than itself. Now if we consider a case where the three Cartesian axes are aligned along the principal direction of the crsytal, we can write:

$$D_x = \vec{D}\cdot\hat{x} = \epsilon_x E_x = n_x^2\epsilon_0 E_x \tag{37}$$

etc. for y and z.

Then it can be shown that:

$$\frac{1}{n^2} = \frac{s_x^2}{n^2 - n_x^2} + \frac{s_y^2}{n^2 - n_y^2} + \frac{s_z^2}{n^2 - n_z^2} \tag{38}$$

Thus given the direction of the input wave $\hat{s} = s_x\hat{x} + s_y\hat{y} + s_z\hat{z}$, and the refractive indices along the principal axes n_x, n_y, n_z, we can determine the refractive index n seen by this wave.

One unusual feature now is that the Poynting vector $\vec{S}$ is not parallel to the phase velocity (which is still in the direction of $\hat{s}$).

$\vec{S}$ remains perpendicular to the electric field $\vec{E}$, which is alson no longer perpendicular to the wavevector $\vec{k}$.

5 Note on complex representations

We chose to take a plane wave of the form

$$\vec{E} = \vec{E}_c(\vec{r}, t) = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t) \tag{39}$$

Of course we could easily have chosen:

$$\vec{E}_s(\vec{r}, t) = \vec{E}_0 \sin(\vec{k} \cdot \vec{r} - \omega t) \tag{40}$$

and gotten the same results.

Since Maxwell's equations are linear, any combinations of these is also a solution, with the same properties of the above, so

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \left[\cos(\vec{k} \cdot \vec{r} - \omega t) + i \sin(\vec{k} \cdot \vec{r} - \omega t)\right] \tag{41}$$

($i = \sqrt{-1}$) is also appropriate, enven though it is complex.

The value of using a complex field is that the math is much cleaner, and it is simple to get the real field by taking the real part of the complex field at the end of the calculation.

So we will generally use

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \, e^{i(\vec{k} \cdot \vec{r} - \omega t)} \tag{42}$$

in solving problems.

There are a few useful tricks to learn (which you can work out for yourself), such as the time averaging needed to evaluate the intensity of the radiation. Recall

$$\vec{I} = \langle \vec{S} \rangle = \frac{1}{2T} \int_{-T}^{T} dt \, \vec{S}(t) \tag{43}$$

note

$$\vec{S}(t) = \vec{E}(t) \times \vec{H}(t) \tag{44}$$

(spatial dependence suppressed for clarity). Using complex fields this becomes

$$\begin{aligned} \vec{I} &= \langle \mathbb{R}\mathrm{e}(\vec{E}) \times \mathbb{R}\mathrm{e}(\vec{H}) \rangle \\ &= \frac{1}{2} \mathbb{R}\mathrm{e}(\vec{E} \times \vec{H}^*) \end{aligned} \tag{45}$$

where $\vec{H}^*$ is the complex conjugate of the magnetic complex field.

The time averaging is thus trivial in this notations, since $\vec{E} \propto e^{-i\omega t}$ and $\vec{H}^* \propto e^{i\omega t}$; and the exponential factors cancel