Paper C2: Laser Science and Quantum Information Processing

Problem Sets 2011 - 12

Problems (2011/12): Set 1

1. (a) By considering the growth in spectral intensity of a beam of radiation as it propagates through an inverted medium, or otherwise, show that the optical gain cross-section of a homogeneously broadened laser transition is given by,[1]

$$\sigma_{21}(\omega - \omega_0) = \frac{\hbar\omega}{\mathrm{c}} B_{21} g_{\mathrm{H}}(\omega - \omega_0), \tag{1.1}$$

   where B_{21} is the Einstein B-coefficient, ω_0 the central frequency, and $g_{\mathrm{H}}(\omega - \omega_0)$ the lineshape of the transition.

   (b) A laser operates on a homogeneously broadened transition between upper and lower levels of fluorescence lifetimes τ_2 and $\tau_1$respectively, which are pumped at rates of R_2 and R_1 respectively. Show that the gain coefficient of the transition is reduced by the presence of intense, narrow-band radiation of total intensity I and frequency ω_{L}, according to:

$$\alpha_I(\omega - \omega_0) = \frac{\alpha_0(\omega - \omega_0)}{1 + I/I_{\mathrm{s}}}, \tag{1.2}$$

   where $\alpha_0(\omega - \omega_0)$ is the small-signal gain coefficient, and the saturation intensity is given by,

$$I_{\mathrm{s}} = \frac{\hbar\omega_0}{\sigma_{21}(\omega_{\mathrm{L}} - \omega_0)\tau_{\mathrm{R}}.} \tag{1.3}$$

   (c) Show that the recovery time τ_{R}, is given by,

$$\tau_{\mathrm{R}} = \tau_2 + \frac{g_2}{g_1}\tau_1 \left[1 - A_{21}\tau_2\right], \tag{1.4}$$

   where g_2 and g_1 are the degeneracies of the upper and lower levels respectively.

2. (a) Explain what is meant by homogeneous broadening and inhomogeneous broadening of a laser transition, and give two examples of each type.

   (b) Discuss briefly the differences between the behaviour of lasers operating on homogeneously- and inhomogeneously-broadened transitions as they are brought above threshold.

[1] For similar problems try the following old Paper B2 Finals questions: Q4 2004, Q4 2002, Q1 2001, Q3 2000 (excluding part on dye lasers).

(c) A continuous wave He-Ne gas laser is operated at 632.8 nm in a single transverse mode. The cavity consists of two mirrors separated by spacer bars of length L. The spacer bars have a linear expansion coefficient of 1×10^{-5} K^{-1}. After the laser is switched on the spacer bars slowly increase in temperature. For the laser operating at a constant but low discharge current, the laser oscillates periodically producing a short burst of output every 10 seconds. During each burst, the laser output at first increases to a peak intensity then decreases to zero and there is no detectable r.f. component in the output signal.

The discharge current is now increased very slightly so that the output is present continuously although the intensity fluctuates with a period of 10 seconds. An r.f. component at 300 MHz is periodically present in the laser output. Account for these observations and calculate the cavity length and rate of rise in temperature of the spacer bars.

3. This problem deals with a few straightforward properties of Fourier transforms. It is adapted from the 1990 Finals paper.

The Fourier transform $\tilde{V}(\omega)$ of a function $V(t)$ is defined by

$$V(t) = \int_{-\infty}^{\infty} \tilde{V}(\omega) e^{-i\omega t} \frac{d\omega}{2\pi}. \tag{1.5}$$

(a) Show that $\tilde{V}^*(\omega) = \tilde{V}(-\omega)$ for the case that $V(t)$ is real.

(b) Prove Parseval's theorem in the form

$$\int_{-\infty}^{\infty} |V(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{V}(\omega)|^2 \frac{d\omega}{2\pi}. \tag{1.6}$$

for the case where $V(t)$ is complex.

(c) Real functions $f(x), g(x)$ possess Fourier transforms $F(k), G(k)$. Show that:

i. $F^*(k)$ is the transform of $f(-x)$

ii. $F(k).G(k)$ is the transform of the convolution $\int f(x')g(x-x')dx'$.

iii. $F(k).G^*(k)$ is the transform of the correlation $\int f(x')g(x'-x)dx'$.

iv. $f(x-a)$ is the inverse Fourier transform of $F(k)e^{-ika}$.

4. Determine the Fourier transform of each of the following functions $f(x)$.

(a) $f(x) = 0$ for $x < -L/2, x > L/2$
$f(x) = A\cos(k_0 x)$ for $-L/2 \le x \le L/2$

(b) $f(x) = 0$ for $x < -L/2, x > L/2$
$f(x) = A\sin^2(k_0 x)$ for $-L/2 \le x \le L/2$

(c) $$f(x) = \sum_{n=0}^{N-1} g(x - nc)$$

where $g(x) = 1$ for $-\frac{1}{2}(b+a) < x < -\frac{1}{2}(b-a)$
$g(x) = 1$ for $\frac{1}{2}(b-a) < x < \frac{1}{2}(b+a)$
$g(x) = 0$ otherwise

5. The operation of a CD or DVD player is predicated on the characteristics of a laser beam that is reflected from the disc, and undergoes diffraction as it propagates towards the detector. This problem asks you to calculate the form of the diffraction pattern, although in one dimension rather than two. The disc is constructed so that a circular laser beam of radius $r > d$ is incident on a "pixel" of radius d. The

diffracted beam under this condition represents, say, a logical "0". A logical "1" is encoded by a pit of radius $d/2$ centred in the pixel. The diffracted beams are collected by a lens and imaged on to a detector. The aperture of the lens is chosen so that a reasonable contrast is possible between the "0" and "1". This problem is adapted from Finals 1991.

(a) Explain the physical basis for expressing the amplitude of a diffracted light wave by an integral of the Kirchoff form.

(b) A beam of light of wavelength $\lambda = 2\pi/k$ is collimated to travel as a plane wave in the z-direction. It falls on a slit of width $4d$ lying in the xy plane and at a large distance. Show that the diffracted amplitude is given by

$$U_0 4d \frac{\sin(2kd\sin(\theta))}{(2kd\sin(\theta))}. \tag{1.7}$$

where U_0 is a constant and θ is the angle between the direction of observation and the z-axis.

(c) The central $2d$ of the slit is next covered with a transparent film which has the effect of phase-delaying the light passing through it by π relative to that passing through the outer portions of the slit. Show that the amplitude diffracted in this case is

$$-8U_0 d \frac{\sin(kd\sin(\theta))}{(kd\sin(\theta))} \sin^2\left(\frac{1}{2}kd\sin(\theta)\right). \tag{1.8}$$

(d) Sketch the intensity as a function of θ, for this case and for the case where the slit is unobstructed.

(e) A device for detecting the presence or absence of the film consists of a lens that collects light transmitted by the slit and delivers it to a photodetector of large area. The radius of the lens subtends an angle $\alpha = \sin^{-1}(\lambda/4d)$ at the slit. Comment on the suitability of the lens for this purpose.

6. (a) Given that for an ideal four-level laser the rate equations for the population inversion density $N^*(t)$ and the photon density $n(t)$ may be written as:

$$\frac{dN^*}{dt} = R_2 - N^*\sigma_{21}\frac{I}{\hbar\omega} - \frac{N^*}{\tau_2} \tag{1.9}$$

$$\frac{dn}{dt} = f_{\mathrm{c}}\sigma_{21}\mathrm{c}N^*n - \frac{n}{\tau_{\mathrm{c}}}, \tag{1.10}$$

where the symbols are defined in the lecture notes, show that under steady-state conditions the population inversion and photon densities are given by,

$$N_{\mathrm{th}^*} = \frac{1}{f_{\mathrm{c}}\sigma_{21}c\tau_{\mathrm{c}}} \tag{1.11}$$

$$n_0 = (r-1)\frac{N^*_{\mathrm{th}} f_{\mathrm{c}}\tau_{\mathrm{c}}}{\tau_2}, \tag{1.12}$$

where $r = R_2\tau_2/N^*_{\mathrm{th}}$ is the over-pumping ratio.

(b) By considering small departures, ΔN^* and Δn, from these equilibrium values, show that the rate equations for the population inversion and photon densities may be linearized to:

$$\frac{d\Delta N^*}{dt} = -r\frac{\Delta N^*}{\tau_2} - \frac{1}{f_{\mathrm{c}}}\frac{\Delta n}{\tau_{\mathrm{c}}} \tag{1.13}$$

$$\frac{d\Delta n}{dt} = (r-1)f_{\mathrm{c}}\frac{\Delta N^*}{\tau_2}. \tag{1.14}$$

(c) Hence, by assuming solutions of the form $\Delta N^*(t) = a \exp(mt)$, $\Delta n(t) = b \exp(mt)$, show that no relaxation oscillations will occur if,

$$\tau_2 < \frac{r^2}{r-1}\frac{\tau_c}{4}. \tag{1.15}$$

(d) Consider laser oscillation in a semiconductor diode lasers. In such systems the cavity is formed by the Fresnel reflections from the cleaved end faces of the semiconductor crystal, and has a length typically equal to 250 μm. If the fluorescence lifetime of the upper laser level is 1 ns, and the refractive index of the semiconductor is equal to 3.6, discuss whether it is possible that relaxation oscillations will occur.

7. (a) What is meant by the term *Q-switching*?[2] Include in your discussion sketches of the gain, loss and photon flux as functions of time for a Q-switched laser, as well as a description of the elements needed for such a laser, and a sketch of the cavity layout.

(b) The rate equations for the population inversion density and photon density during the Q-switched pulse may be written:

$$\frac{dN^*}{dt} = -\beta N^* \sigma_{21}\frac{I}{\hbar\omega} \tag{1.16}$$

$$\frac{dn}{dt} = \left(\frac{N^*}{N^*_{\text{th}}} - 1\right)\frac{n}{\tau_c}. \tag{1.17}$$

Show that the energy that is extracted in the Q-switched pulse is given by,

$$E = \eta N_i^* V_g \hbar\omega, \tag{1.18}$$

where V_g is the volume of the gain region and the *energy utilization factor* is given by,

$$\eta = \frac{N_i^* - N_f^*}{\beta N_i^*}. \tag{1.19}$$

Interpret this last result for the cases of: (i) an ideal four-level laser; (ii) severe bottle-necking.

(c) Show that the photon density at the peak of the Q-switched pulse is given by,

$$n_{\text{peak}} = \frac{f_c}{\beta} N^*_{\text{th}} \left[r - 1 - \ln r\right]. \tag{1.20}$$

(d) Hence derive an approximate expression for the duration of the Q-switched laser pulse, and explain physically the conditions under which the output pulse duration tends to the cavity decay time.

(e) A laser cavity consists of two mirrors, having reflectivities $R_1 = 0.98$ and $R_2 = 0.80$. The cavity length is 0.3 m. The saturation fluence of the gain medium, $\hbar\omega/2\sigma_{21}$ is 1 J cm^{-2} and the beam diameter throughout the gain medium is approximately 1 mm. Assuming that no bottlenecking occurs, calculate the energy, peak power and pulse duration from this laser when Q-switched with an initial inversion density 500 times above threshold.

[2] For similar Finals questions see: Q5 C2 2007 and Q4 C2 2006. Also try the following old Paper B2 Finals questions: Q1 2003, Q2 2002.

Problems (2011/12): Set 2

1. (a) Discuss briefly what is meant by the term *modelocking* and explain why it is a useful technique.
   (b) Discuss what is meant by *active* and *passive* modelocking, and describe briefly an example of each type.

2. Here we consider a more general case of modelocking in which the frequencies and phases of the longitudinal modes are given by[1]

$$\begin{aligned} \omega_n &= n\Delta\omega + \delta\omega && (2.1) \\ \phi_n &= n\Delta\phi + \delta\phi && (2.2) \end{aligned}$$

where n is an integer, and $\Delta\omega$ the frequency between adjacent modes.

   (a) Suppose that the laser oscillates on an odd number of modes labelled $n = P_0 - P/2 \ldots P_0 + P/2$. Show that at the point $z = 0$ the amplitude of the beam emitted from the laser will have the form,

$$\begin{aligned} E(0,t) &= \exp\left(-\mathrm{i}[P_0\Delta\omega t + \delta\omega t - P_0\Delta\phi - \delta\phi]\right) \\ &\times \sum_{p=-P/2}^{p=+P/2} a_p \exp\left(-\mathrm{i}p[\Delta\omega t - \Delta\phi]\right). \qquad (2.3) \end{aligned}$$

   where a_p is the amplitude of the mode with $n = P_0 + p$.

   (b) Identify the factors corresponding to the carrier wave and pulse envelope in the above result. Show that the peaks of the envelopes of the pulses occur at times given by,

$$t_q = \frac{2\pi}{\Delta\omega}q + \frac{\Delta\phi}{\Delta\omega} \quad q = 0, 1, 2, 3, \ldots \qquad (2.4)$$

   and hence that the phase difference $\Delta\phi$ causes the peaks of the pulse envelopes to shift in time by an amount $\Delta\phi/\Delta\omega$ compared to the case when $\Delta\phi = 0$.

   (c) Show that the **carrier-envelope offset phase**, that is the phase of the carrier wave at the peaks in the pulse envelope is given by,

$$\phi_{\mathrm{CEO}} = 2\pi\left[P_0 q + q\frac{\delta\omega}{\Delta\omega} + \frac{\delta\omega}{\Delta\omega}\frac{\Delta\phi}{2\pi} - \frac{\delta\phi}{2\pi}\right]. \qquad (2.5)$$

   (d) Hence show that the phase of the carrier wave at the peak of the pulse envelope will change unless $\delta\omega = 0$. When might control of ϕ_{CEO} be important?

[1] For more modelocking questions, try the following old Paper B2 Finals question: Q1 2004.

(e) We now consider the case of a Gaussian spectrum of longitudinal modes. (For the remainder of the question assume that $\Delta\phi = \delta\phi = \delta\omega = 0$.) Suppose that the amplitudes of the oscillating modes are described by,

$$a_p = \exp\left[-\left(\frac{p\Delta\omega}{\Delta\omega'}\right)^2\right]. \tag{2.6}$$

Find the full-width at half maximum, $\Delta\omega_{\mathrm{FWHM}}$, of the *power* spectrum in terms of $\Delta\omega'$ (assume $\Delta\phi = 0$).

(f) By approximating the sum in eqn (2.3) to an integral, show that the modelocked pulse has a Gaussian envelope in time, and find the full-width at half maximum, Δt_{FWHM}, of the *intensity* profile. [You may require the identity $\int_{-\infty}^{\infty} \exp(-a^2\omega^2)\exp(-\mathrm{i}\omega t)d\omega = \sqrt{(\pi/a)}\exp(-t^2/4a^2)$.]

(g) Hence show that the **time-bandwidth product** of the modelocked pulse obeys,

$$\Delta\omega_{\mathrm{FWHM}}\Delta t_{\mathrm{FWHM}} = 4\ln 2. \tag{2.7}$$

(h) What bandwidth (in cm^{-1}) would be needed to generate a pulse of 10 attosecond duration, and what could be the longest mean wavelength of such a pulse? [1 attosecond = 10^{-18} s]

3. (a) What is meant by the terms *group delay dispersion* and *frequency chirp*?

(b) Discuss the importance of *dispersion control* in the production, amplification, and propagation of short laser pulses.

(c) We now derive the basic properties of a Gaussian optical pulse described by

$$E_{\mathrm{in}}(t) = \mathrm{e}^{-\Gamma t^2}\mathrm{e}^{-\mathrm{i}\omega_0 t} \tag{2.8}$$

where the complex Gaussian parameter is defined by $\Gamma \equiv a + \mathrm{i}b$. Show that the full-width at half maximum duration of the *intensity* profile of the pulse is given by

$$\tau_{\mathrm{p}} = \sqrt{\frac{2\ln 2}{\Re(\Gamma)}} = \sqrt{\frac{2\ln 2}{a}}.$$

(d) Given the standard integral,

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\beta t^2}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t = \sqrt{\frac{\pi}{\beta}}\mathrm{e}^{-\omega^2/4\beta} \qquad \text{if } \Re(\beta) > 0, \tag{2.9}$$

calculate the Fourier transform of eqn (2.8) to show that the amplitude per unit frequency interval is given by

$$a(\omega') = \sqrt{\frac{1}{2\Gamma}}\mathrm{e}^{-\omega'^2/4\Gamma}, \tag{2.10}$$

where $\omega' = \omega - \omega_0$.

(e) Show that the full-width at half maximum width of the *power* spectrum is given by

$$\Delta\omega_{\mathrm{p}} = 2\sqrt{2\ln 2}\sqrt{a[1 + (b/a)^2]},$$

and hence show that the time-bandwidth product of the Gaussian pulse is equal to

$$\Delta\omega_{\mathrm{p}}\tau_{\mathrm{p}} = 4\ln 2\sqrt{1 + (b/a)^2}.$$

4. We now consider the propagation of the Gaussian optical pulse through a dispersive medium in which the accumulated phase is given by

$$\phi(\omega) = \phi^{(0)} + \phi^{(1)}(\omega - \omega_0) + \frac{1}{2}\phi^{(2)}(\omega - \omega_0)^2.$$

(a) By considering the propagation of each frequency component, show that the amplitude of the electric field at the end of the medium is given by,

$$\begin{aligned} E_{\mathrm{out}}(t) = & \sqrt{\frac{1}{4\pi\Gamma_{\mathrm{in}}}} \mathrm{e}^{\mathrm{i}[\phi^{(0)} - \omega_0 t]} \\ & \times \int_{-\infty}^{\infty} \exp\left[-\left(\frac{1}{4\Gamma_{\mathrm{in}}} - \mathrm{i}\frac{\phi^{(2)}}{2}\right)\omega'^2\right] \\ & \times \exp\left\{-\mathrm{i}\omega'\left[t - \phi^{(1)}\right]\right\} \mathrm{d}\omega'. \end{aligned}$$

(b) Using eqn (2.9), or by comparing the above integral to eqn (2.10), show that the amplitude of the transmitted pulse is given by

$$E_{\mathrm{out}}(t) = \sqrt{\frac{\Gamma_{\mathrm{out}}}{\Gamma_{\mathrm{in}}}} \exp[\mathrm{i}(\phi^{(0)} - \omega_0 t)] \exp\left\{-\Gamma_{\mathrm{out}}\left[t - \phi^{(1)}\right]^2\right\},$$

where

$$\frac{1}{\Gamma_{\mathrm{out}}} = \frac{1}{\Gamma_{\mathrm{in}}} - 2\mathrm{i}\phi^{(2)}. \tag{2.11}$$

(c) We now consider the form of the Gaussian pulse as a function of the distance z it propagates through a quadratically-dispersive medium. Use eqn (2.11) to show that the real and imaginary parts of the Gaussian parameter $\Gamma(z) = a(z) + \mathrm{i}b(x)$ are given by

$$a(z) = \frac{a_0}{[1 + 2b_0\phi^{(2)}]^2 + [2a_0\phi^{(2)}]^2} \tag{2.12}$$

$$b(z) = \frac{b_0 + 2\phi^{(2)}(a_0^2 + b_0^2)}{[1 + 2b_0\phi^{(2)}]^2 + [2a_0\phi^{(2)}]^2} \tag{2.13}$$

where a_0 and b_0 are the real and imaginary parts of Γ at $z = 0$.

(d) Show that the duration of the transmitted pulse is given by

$$\tau_{\mathrm{p}}(z) = \tau_{\mathrm{p}}(0)\sqrt{\left[1 + 2b_0\phi^{(2)}\right]^2 + \left[2a_0\phi^{(2)}\right]^2}. \tag{2.14}$$

(e) Equation (2.14) shows that the duration of the pulse always increases with z if $\mathrm{b}_0\phi^{(2)} \geq 0$. Explain qualitatively why this occurs.

(f) Sketch the behaviour of $\tau_{\mathrm{p}}(z)$ for the case $\mathrm{b}_0\phi^{(2)} > 0$ and $\mathrm{b}_0\phi^{(2)} < 0$.

(g) Discuss how the bandwidth of the pulse varies as it propagates through the medium.

5. (a) Show that in propagating through a length L of material with non-linear refractive index n_2 an unchirped, intense pulse will develop a time-dependent angular frequency given by,

$$\omega'(t) = \omega_0 - \frac{2\pi}{\lambda_0} n_2 L \frac{\partial I}{\partial t}, \tag{2.15}$$

where ω_0 and λ_0 are the central frequency and wavelength of the radiation.

(b) Hence show that a pulse with a Gaussian temporal profile of the form $I(t) = I_0 \exp(-2\ln 2[t/\Delta t]^2)$, where Δt is the full-width at half maximum, develops an additional frequency bandwidth given by,

$$\Delta\omega = 2B\frac{2}{\Delta t}\sqrt{\frac{\ln 2}{e}}, \tag{2.16}$$

where B is the value of the B-integral at the peak of the laser pulse.

(c) A Ti:sapphire laser delivers pulses of duration $\Delta t = 50\,\mathrm{fs}$ with a peak power of 2 TW. Taking the diameter of the beam to be 10 mm, and assuming a mean wavelength of 800 nm, calculate the B-integral in:

i. Propagating through a 10 mm thick silica window, and estimate the bandwidth of the transmitted pulse;

ii. Propagating through 1 m of air.

[For silica $n_2 = 2.73 \times 10^{-20}\ \mathrm{m^2\,W^{-1}}$; for air $n_2 = 6.3 \times 10^{-23}\ \mathrm{m^2\,W^{-1}}$]

(d) Discuss briefly the implication of your results for manipulating TW laser beams.

6. The diffraction from an aperture becomes a Fourier transform of the transfer function of the aperture under several different circumstances. This problem explores the possible configurations that realise this situation.

(a) Use the Fresnel approximation to the Kirchoff diffraction integral to calculate the scalar field at a plane (the image plane) a distance v from a thin lens of focal length f, which is in turn a distance u from the aperture plane.

(b) The aperture is illuminated with a plane wave of wavelength $\lambda = 2\pi/k$. Show that the diffracted beam in the image plane is a Fourier Transform of the aperture function under conditions:

i. $u = 0, v = f$ (i.e. the aperture is at the lens)

ii. $u = f, v = f$ (i.e. the aperture is at the back focal plane of the lens)

iii. $u = -d, v = f$ (i.e. the aperture is between the lens and the image plane. In this case the plane wave is incident on the lens.)

In each case sketch the setup and determine the scale factors that specify the relationship between the coordinate in the image plane and the corresponding transverse wavenumber in the aperture function. In which cases is the Fourier transform relationship exact?

7. Consider a Gaussian beam of wavelength λ propagating along the z-axis, towards positive z, focused by a thin lens of focal length f located at $z = 0$. Let us suppose that the Rayleigh ranges (or confocal parameters) of the beam are b_1 and b_2 in the regions $z < 0$ and $z > 0$ respectively.

   (a) Let the beam have a waist at $z = -z_1$. What is the complex radius of curvature q at this point? Write down the ray transfer matrix for translation through a distance L, and use this to show that for $z < 0$ the complex radius of curvature is given by,

   $$q_1(z) = z + z_1 + \mathrm{i}b_1. \tag{2.17}$$

   (b) Write down the ray transfer matrix for a thin lens, and use this to find an expression for $q_2 = z - z_2 + \mathrm{i}b_2$, the complex radius of curvature in the region $z > 0$.

   (c) Show that,

   $$z_2 = -\frac{z_1\left(1 - z_1/f\right) - b_1^2/f}{\left(1 - z_1/f\right)^2 + \left(b_1/f\right)^2} \tag{2.18}$$

   $$b_2 = \frac{b_1}{\left(1 - z_1/f\right)^2 + \left(b_1/f\right)^2}. \tag{2.19}$$

   (d) Using the relation between $q_1(0)$ and $q_2(0)$ found from the ray transfer matrix show that,

   $$\frac{1}{R_1(0)} - \frac{1}{R_2(0)} = \frac{1}{f} \tag{2.20}$$

   $$w_1(0) = w_2(0) \tag{2.21}$$

   where $R_1(0)$ and $R_2(0)$ are the radii of curvature of the wavefronts, and $w_1(0)$ and $w_2(0)$ the spot sizes, immediately before and after the lens.

   (e) We now consider the limit $b_1/f \ll 1$.

      i. Show that in this case:

      $$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f} \tag{2.22}$$

      $$\frac{w_2}{w_1} = \frac{z_2}{z_1}, \tag{2.23}$$

      where $w_i = \sqrt{\lambda b_i/\pi}$ is the spot size at the beam waist.

      ii. Comment on the results obtained in this limit, and sketch the propagation of the rays through the system.

   (f) We now consider the opposite limit: $b_1/f \gg 1$.

      i. Show that in this case:

      $$z_2 = f \tag{2.24}$$

      $$w_2 = \frac{\lambda}{\pi w_1} f. \tag{2.25}$$

ii. Show that the size of the beam waist is approximately the same as the spot size predicted by scalar diffraction theory for a plane wave propagating through a circular aperture of radius w_1.

iii. Comment on the results obtained in this limit, and sketch the propagation of the rays through the system.

Problems (2011/12): Set 3

1. (a) Explain why a double heterojunction structure helps to improve the output characteristics of a semiconductor laser.

   (b) The small-signal gain coefficient $\alpha(\omega)$ of a semiconductor device is found to vary with frequency as follows:

   $$\alpha(\omega) = K\left[f_c(E_2) - f_v(E_1)\right](\hbar\omega - E_g)^{1/2}, \tag{3.1}$$

   where E_g is the band gap energy. Interpret this equation, explaining in particular the power dependence of the laser frequency and the origin of any temperature dependence.

   (c) A GaAs laser has carrier concentrations of $10^{24}\ \mathrm{m}^{-3}$ for both holes and electrons. The effective masses are $m_v = 0.1m_e$ and $m_c = 0.07m_e$ for carriers in the valence and conduction bands, respectively. Calculate the maximum value of the gain coefficient at 0 K and the frequency at which it occurs.

   [At a temperature of 0 K the band gap energy of GaAs is 1.512 eV and the factor K is $6 \times 10^5\mathrm{m}^{-1}\,\mathrm{eV}^{-1/2}$]

2. Here we evaluate the key parameters involved in determining the Schawlow-Townes linewidth in a He-Ne laser. Suppose that the laser operates on a single longitudinal mode, that it delivers 1 mW on the 632.8 nm line, and that the cavity is 30 cm long cavity with $R_1 = 1$, $R_2 = 0.98$.

   (a) Find:

      i. The cavity lifetime τ_c.

      ii. The cavity linewidth $\Delta\nu_c$.

      iii. The Schawlow–Townes linewidth $\Delta\nu_{ST}$.

      iv. The coherence time τ_{ST} against dephasing by spontaneous emission into the oscillating mode.

      v. The mean number of photons $\bar{n}$ in the oscillating mode.

   (b) Calculate the change in cavity length which would shift the frequency of the oscillating mode by an amount equal to $\Delta\nu_{ST}$.

   (c) Suppose that the laser cavity is mounted a support made from invar. How stable would the temperature need to be controlled to keep drifts in the laser frequency to within $\Delta\nu_{ST}$? (The coefficient of thermal expansion of invar is approximately 1×10^{-6}.)

3. To gain some insight into the operation of waveguides, we first consider the propagation of EM waves in a simple, 1D waveguide comprised of two infinite, perfectly conducting plates separated by a distance d, as shown in Fig. 3.1.[1] For the moment we will consider the case when the electric field is polarized parallel

[1]Further discussions of the theoretical treatment of waveguides may be found in Yariv "Optical Electronics in Modern Communications" p. 491 and Davis, "Lasers and Electro-Optics" p. 395.

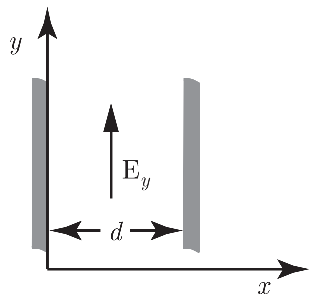

Figure 3.1: A 1D waveguide formed by two infinite, conducting plates.

to the x-axis; the polarization of the magnetic field will remain unspecified, and will need to be found. We will try and find solutions which are close to, but not quite, the plane wave solutions we are used to seeing, i.e. we look for solutions of the form:

$$\boldsymbol{E}(\boldsymbol{r},t) = E_y\boldsymbol{j} = E_0(x)\boldsymbol{j}\exp\left[\mathrm{i}(\beta z - \omega t)\right] \tag{3.2}$$
$$\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{H_0}(x)\exp\left[\mathrm{i}(\beta z - \omega t)\right], \tag{3.3}$$

where ω is the angular frequency of the waves and the propagation constant β is to be found.

(a) Describe the ways in which these trial solutions differ from that of a plane wave.

(b) Given that the plates have infinite extent in the y-direction, show that the Maxwell equation $\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$ yields:

$$-\beta E_y = \mu\omega H_x \tag{3.4}$$
$$0 = H_y \tag{3.5}$$
$$\frac{\partial E_y}{\partial x} = \mathrm{i}\mu\omega H_z \tag{3.6}$$
$$\tag{3.7}$$

(c) Similarly, show that the Maxwell equation $\nabla \times \boldsymbol{H} = \partial \boldsymbol{D}/\partial t$ yields:

$$\frac{\partial H_z}{\partial x} - \mathrm{i}\beta H_x = \mathrm{i}\epsilon\omega E_y. \tag{3.8}$$

(d) Eliminate H_x to show that E_y satisfies:

$$\frac{\mathrm{d}^2 E_0}{\mathrm{d}x^2} = -\mathrm{k}_x^2 E_0, \tag{3.9}$$

where

$$k_x^2 = \mu\epsilon\omega^2 - \beta^2. \tag{3.10}$$

(e) State the boundary condition which must be satisfied by the electric field at the two plates, and hence show that,

$$E_0 = A \sin\left(\frac{\pi}{d} p x\right) \qquad p = 1, 2, 3, \ldots, \tag{3.11}$$

where A is a constant.

(f) Using this result, derive the *dispersion relation* for the waveguide:

$$\beta^2 = \mu\epsilon\omega^2 - \left(\frac{\pi p}{d}\right)^2, \tag{3.12}$$

and comment on this result.

(g) Using your solution for $\boldsymbol{E}$ and eqns (3.4) - (3.8) to find the magnetic field $\boldsymbol{H}$.

(h) Show that in order for the wave to propagate along the waveguide the frequency of the wave must exceed the cut-off frequency ω_c, and find an expression for ω_c.

(i) Use the dispersion relation to show that:

$$v_\mathrm{p} = \frac{c}{n}\left[1 - \left(\frac{\omega_c}{\omega} p\right)^2\right]^{-1/2} \tag{3.13}$$

$$v_\mathrm{p} v_\mathrm{g} = \left(\frac{c}{n}\right)^2, \tag{3.14}$$

where the phase and group velocities of the wave are $v_\mathrm{p} = \omega/\beta$ and $v_\mathrm{g} = \partial\omega/\partial\beta$ respectively, and c/n is the velocity of EM waves in an unbounded medium made of the same material as that between the plates. On the same graph, sketch v_p and v_g as a function of ω/ω_c for the mode $p = 1$.

(j) The set of solutions, labelled by the parameter p, form the a set of *waveguide modes.* Outline the ways in which these modes differ from the plane wave solutions found in an unbounded medium.

4. This problem deals with slab waveguides. The treatment is relevant for waves in all sorts of non-isotropic media, and the elements needed for determining the conditions for guiding are of particular importance. Consider a 2-dimensional structure consisting of three layers of dielectric. In the vertical (x) direction, transverse to the direction of wave propagation (z), there is a substrate (refractive index n_s), extending from $x = -\infty$ to $x = 0$, an intermediate layer, (n_f, between $x = 0$ and $x = h$) and a cover, (n_c, between $x = h$ and $x = +\infty$). The task is to find the existence condition for guided waves by deriving a dispersion relation between the frequency and wavevector, and to determine the number of waves that can propagate in the structure.

   As in the previous question we postulate a transverse electric (TE) form for the wave fields, with linear polarization in the y direction. Specifically:

$$\begin{aligned}
E(x, y, z) &= \hat{y}\; C\; e^{(i\beta z - i\omega t)}\; e^{-\gamma_c (x-h)} \quad \text{for} \;\; x > h, \\
E(x, y, z) &= \hat{y}\; [A e^{i\kappa_f x} + B e^{-i\kappa_f x}]\; e^{(i\beta z - i\omega t)} \quad \text{for} \;\; 0 \le x \le h, \\
E(x, y, z) &= \hat{y}\; S\; e^{(i\beta z - i\omega t)} e^{+\gamma_s x} \quad \text{for} \;\; x < 0.
\end{aligned} \tag{3.15}$$

(a) Comment on the form of these trial solutions. What are the boundary conditions that must be satisfied by the transverse components of $\boldsymbol{E}$ and $\boldsymbol{H}$? Use these to find relationships between the field amplitudes S, C, A, and B; eliminate these amplitudes to find the following dispersion relation:

$$\kappa_f h - \arctan\left(\frac{\gamma_c}{\kappa_f}\right) - \arctan\left(\frac{\gamma_s}{\kappa_f}\right) = m\pi. \tag{3.16}$$

(b) Show that the field in the waveguide center layer is

$$E(x,y,z) = \hat{y}\; E_f \cos(\kappa_f x - \phi_s)\; e^{(i\beta z - i\omega t)}, \tag{3.17}$$

where $\phi_s = \arctan(\frac{\gamma_s}{\kappa_f})$, and $E_f = 2A\; e^{i\phi_s}$.

(c) Determine the field amplitudes C and S in terms of E_f and other parameters.

(d) The guide only supports certain *modes*, since the *cut-off* condition, $\gamma_s = 0$ sets a bound on which waves are actually guided in the structure. Show that this implies, for a symmetric guide, when $m = 0$, that

$$\frac{2\pi h}{\lambda_0}\sqrt{n_f^2 - n_s^2} = \arctan\sqrt{a}, \tag{3.18}$$

where a is the asymmetry parameter for the guide

$$a = \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2}. \tag{3.19}$$

[Hint: you may need to use the relations between γ_s, γ_c, κ_f and β derived from the fact that the fields must satisfy the wave equation.]

(e) Show that this implies that a symmetric guide ($a = 0$) will support a mode of any wavelength no matter what thickness.

(f) Show further that the minimum guide thickness that will support a mode with index m is

$$h = \frac{\lambda_0}{2\pi\sqrt{n_f^2 - n_s^2}}(\arctan\sqrt{a} + m\pi). \tag{3.20}$$

Prove that the second term in parentheses is usually significantly larger than the first (use realistic numbers for the refractive indeces of glass), and therefore that the number of modes supported by a structure of given thickness is

$$m = \frac{2h\sqrt{n_f^2 - n_s^2}}{\lambda_0} \tag{3.21}$$

(g) Determine the minimum and maximum thickness layers that will support a single mode for a guide consisting of a substrate layer of quartz ($n_s = 1.47$), a guide layer of glass ($n_f = 1.62$) with air as a cover layer. For a layer thickness halfway between the extremal values, calculate the propagation constant β of the guided wave. How does this compare with the magnitude of the wavevector of a wave of the same frequency in bulk glass of the same refractive index as the guide layer?

5. (a) Explain briefly why crystalline materials are necessary for the observation of the Linear Electro-optic Effect.[2]

[2] For a similar problem see the following old Paper B2 Finals question: Q9 2001.

(b) The LEO tensor for potassium di-deuterium phosphate (KD*P) is

$$r_{LEO} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{52} & 0 \\ 0 & 0 & r_{63} \end{pmatrix}$$

A laser beam is incident normally on one face of a KD*P crystal. The beam propagates along the z-axis of the crystal and is plane-polarised at 45^0 to the x- and y- crystal axes. Find an expression for the change in refractive index seen by the laser beam when an electric field E_z is applied along the z-axis.

(c) When the exit face of the crystal is cut at an angle of 36^0 to the incident face and the field E_z is 10^6 Vm^{-1} the emerging beam is deflected by an additional angle of 0.173 minutes. How may this be explained? Use the information to calculate the value of r_{63} for KD*P. What are the limitations of this device as a beam deflector? How may a more practical device be constructed using the LEO effect? The refractive index for the ordinary ray $n_0 = 1.51$.

6. (a) Some substances have a non-linear relationship between electric polarization and electric field within the material. Explain how this property can be used for generating new optical frequencies.[3]

(b) A light beam of high power and wavelength 694 nm is passed through a slab of crystalline KH_2PO_4 (KDP) at right angles to the optic axis. A wave is generated at the second harmonic of the input frequency. The input wave is polarized with its electric vector at right angles to the optic axis and the second harmonic wave has its electric vector along the optic axis. It is found that the intensity of the second harmonic wave does not always increase with an increase of thickness of the crystal. Give an explanation of this effect and find an estimate of the crystal thickness below which the intensity is an increasing function of thickness.

(c) Comment on the order of magnitude of your calculated thickness. What is the consequence for the second harmonic output when a much thicker crystal is used? Describe an arrangement by which a thick KDP crystal can be used to give efficient generation of the second harmonic.

[The following refractive indices apply to ordinary (o) and extraordinary (e) waves in KDP: n_0 (694nm) = 1.506, n_0 (347nm) = 1.534, n_e (694nm) = 1.466, n_e (347nm) = 1.487.]

7. (a) What is the Pockels effect and how may it be used to modulate the amplitude of a light beam?

(b) When an electric field is applied in the y-direction (the optical axis is in the z-direction) of a lithium niobate crystal, the refractive indices for light whose electric vector is in the x- and y-direction are given respectively by

$$n_x = n_o + \alpha E$$
$$n_y = n_o - \alpha E$$

where n_o is the zero-field refractive index and α has the value 1.6 x 10^{-11} m V^{-1} at 633 nm. In the arrangement you chose what is the magnitude of the voltage that must be available if the modulator is to achieve 100% modulation (i.e. to change its transmission from 100% to zero) using light from a He-Ne laser? The crystal has dimensions along the x-, y- and z-directions of 8, 8 and 24 mm respectively.

[3] For a similar problem see the following old Paper B2 Finals question: Q8 2003

(c) Light leaving the lithium niobate crystal is made to pass through a quarter-wave plate whose fast and slow axes are parallel to the x- and y-axes of the lithium niobate. Describe how this affects the operation of the modulator you have designed.