5. Show that a general pure state of a single qubit can be written as

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$$

Find the corresponding density matrix and give a geometric interpretation of θ and ϕ. [5]

Consider the pure state

$$|\psi^{\perp}\rangle = \cos(\theta'/2)|0\rangle + \sin(\theta'/2)e^{i\phi'}|1\rangle$$

where $\theta' = \pi - \theta$ and $\phi' = \pi + \phi$. Show that $|\psi^{\perp}\rangle$ is orthogonal to $|\psi\rangle$, and give a geometric interpretation of the relationship between $|\psi\rangle$ and $|\psi^{\perp}\rangle$. Use state fidelities to show that $|\psi^{\perp}\rangle$ is the state most unlike $|\psi\rangle$, and that the fidelity between the maximally mixed state and any pure state is 1/2. Show that the state obtained by applying a 180°_{y} rotation to $|\psi^{\perp}\rangle$ has a density matrix which is the transpose of $|\psi\rangle\langle\psi|$. [10]

The operation which converts $|\psi\rangle$ to $|\psi^{\perp}\rangle$ is sometimes called the U-NOT gate, and it can be shown that this gate cannot be implemented physically. It can, however, be approximated. One possible approach is to apply a conventional NOT gate, and another is to measure the qubit in the computational basis and then prepare a qubit in the appropriate state. Find the fidelity between each of these gates and an ideal U-NOT gate for the three initial states corresponding to the intersections of the x, y and z-axes with the Bloch sphere. Comment on the *average* and *worst case* fidelities of these two approximate approaches. [10]

(The fidelity of a gate for a given input is defined as the state fidelity between the output of the gate and the desired output.)