Worked answers to C2 2011 paper

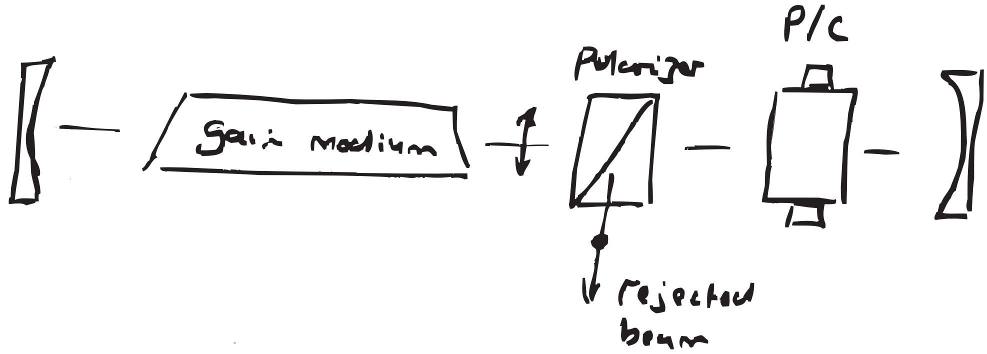
This is a sample set of worked answers to the 2011 paper which would have achieved full marks on all questions. In a few cases we comment briefly on alternative methods of solution which would also have achieved full marks; it would not, of course, be necessary to use more than one approach in real life. Note that it is entirely possible to get full marks for very brief answers if these include all key points.

Question 1: Mode locking

Mode locking [7 marks]

In a mode locked laser several cavity modes are forced to oscillate simultaneously with a constant (possibly zero) phase difference between them. This causes the laser output to comprise a train of short pulses separated by the round-trip time T_c. The duration of each pulse is related to the bandwidth of the oscillating modes; the greater the bandwidth the shorter the pulse can be.

Mode locking can be achieved by modulating the losses of the cavity with a frequency $\Delta\omega$ equal to the frequency spacing of the cavity modes. This can be realized by placing a Pockels cell near one end of the laser cavity as below



With a voltage applied to the Pockels cell it acts as a quarter-wave plate and hence the combination of linear polarizer and Pockels cell rejects the oscillating modes. However if this voltage is removed the Pockels cell does not change the polarization state of the mode and hence the cavity losses are reduced. [Other possible approaches include acousto-optic modulation, Kerr lens mode locking, the use of a saturable absorber, frequency modulation with a Pockels cell or an AOM.]

Electric field [8 marks]

We can represent the (infinitely long) train of pulses separated by T_c as a Fourier series of harmonics of the fundamental frequency $\Delta\omega = 2\pi/T_c$. Hence the field has the given form.

At $z = 0$ we have

$$\begin{aligned} E(0,t) &= \sum_p a_p \mathrm{e}^{-\mathrm{i}[(\omega_{ce}+p\Delta\omega)t-\phi_0-p\Delta\phi]} \\ &= \mathrm{e}^{-\mathrm{i}\omega_{ce}t}\mathrm{e}^{\mathrm{i}\phi_0}\sum_p a_p \mathrm{e}^{-\mathrm{i}p(\Delta\omega t-\Delta\phi)} \end{aligned}$$

Let

$$t' = t - \Delta\phi/\Delta\omega \quad \Rightarrow \quad \Delta\omega t - \Delta\phi = \Delta\omega t'$$

so

$$E(0,t) = \mathrm{e}^{-\mathrm{i}\omega_{ce}(t'+\Delta\phi/\Delta\omega)}\mathrm{e}^{\mathrm{i}\phi_0}\sum_p a_p \mathrm{e}^{-\mathrm{i}p\Delta\omega t'}.$$

Now let $p = p_0 + q$ so

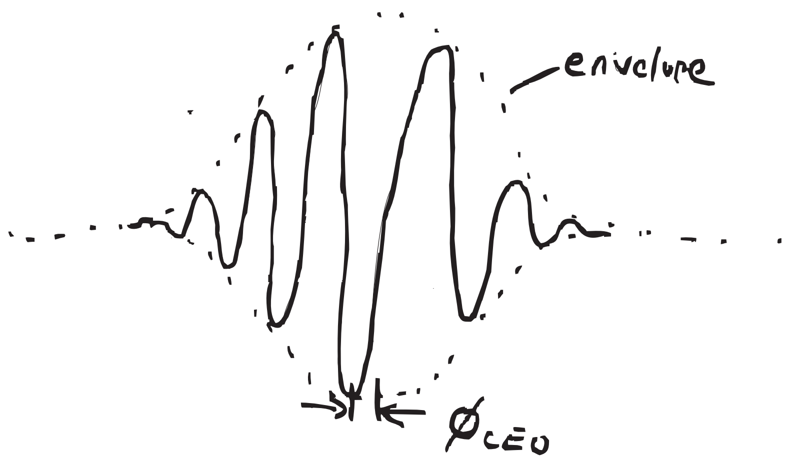
$$\begin{aligned} E(0,t) &= \mathrm{e}^{\mathrm{i}(\phi_0 - \omega_{ce}\Delta\phi/\Delta\omega)}\mathrm{e}^{-\mathrm{i}\omega_{ce}t'}\sum_q a_{p0+q}\mathrm{e}^{-\mathrm{i}(p_0+q)\Delta\omega t'} \\ &= \mathrm{e}^{\mathrm{i}(\phi_0 - (\Delta\phi/2\pi)\omega_{ce}T_c)}\mathrm{e}^{-\mathrm{i}(\omega_{ce}+p_0\Delta\omega)t'} \times \sum_q a_{p0+q}\mathrm{e}^{-\mathrm{i}q\Delta\omega t'} \end{aligned}$$

where the two terms before the $\times$ sign define the carrier wave, and the third term written as

$$\sum_q a_{p0+q}\mathrm{e}^{-\mathrm{i}2\pi q t'/T_c}$$

defines the envelope.

Each pulse has the form



Phase slip [4 marks]

Suppose the envelope has a peak at t'. The next peak will occur at $t' + T_c$, when the field will be

$$E(0,t'+T_c) = \mathrm{e}^{\mathrm{i}(\phi_0 - (\Delta\phi/2\pi)\omega_{ce}T_c)}\mathrm{e}^{-\mathrm{i}(\omega_{ce}+p_0\Delta\omega)(t'+T_c)} \times \sum_q a_{p0+q}\mathrm{e}^{-\mathrm{i}2\pi q(t'+T_c)/T_c}.$$

Now

$$\sum_q a_{p0+q}\mathrm{e}^{-\mathrm{i}2\pi q(t'+T_c)/T_c} = \sum_q a_{p0+q}\mathrm{e}^{-\mathrm{i}2\pi q t'/T_c}$$

and so the envelope is unchanged. However the phase of the carrier-wave changes by

$$|\phi_{\text{slip}}| = (\omega_{ce} + p_0\Delta\omega)T_c = \omega_{ce}T_c + 2\pi p_0$$

Neglecting multiples of 2π

$$|\phi_{\text{slip}}| = \omega_{ce}T_c = 2\pi\omega_{ce}/\Delta\omega.$$

Holding $\phi_{\mathbf{CEO}}$ constant [6 marks]

The value of ϕ_{CEO} is important if there are only a small number of optical cycles within the pulse envelope (a "few cycle" pulse).

The second harmonic component of $p = n$ has a frequency $\omega_1 = 2(\omega_{ce} + n\Delta\omega)$. The component $p = 2n$ has a frequency $\omega_2 = \omega_{ce} + 2n\Delta\omega$. The difference between these two frequencies is

$$\delta = \omega_1 - \omega_2 = 2\omega_{ce} + 2n\Delta\omega - (\omega_{ce} + 2n\Delta\omega) = \omega_{ce}.$$

To keep ϕ_{CEO} fixed we need $\phi_{\text{slip}} = 0$, and hence $\omega_{ce} = 0$. Hence by setting the difference frequency to zero we hold the phase offset ϕ_{CEO} constant.

Question 2: The alexandrite laser

Introduction [9 marks]

Optical pumping on the vibronic transition $^4\text{T} \leftarrow {}^4\text{A}$ (ground state) populates the vibrational levels of $^4\text{T}_2$ and ^2E. Rapid vibrational relaxation leads to a build up of population in the lowest vibrational levels of $^4\text{T}_2$ and ^2E. The population of these levels in maintained in thermal equilibrium by phonon collisions.

Transition α occurs on a zero-phonon transition since there is no change of configuration coordinates. The lower level is the ground state and so the laser operates on a three-level scheme. Since lasing occurs to the ground state the thermal population of the lower laser level is high, and hence it is not usually possible to achieve continuous wave oscillation. Being a zero-phonon transition the laser line has a narrow spectral width.

Transition β occurs on a vibronic transition since there is a change of configuration coordinate. In this case lasing can occur to many vibrational levels of $^4\text{A}_2$ and the laser output can be tuned over a wide frequency range. The lower laser level is an excited vibrational level of $^4\text{A}_2$ which decays rapidly by non-radiative phonon processes. As such the laser can be considered as a four-level scheme, and it is possible to achieve continuous oscillation.

Populations [4 marks]

The populations in levels 3 and 4 are related by

$$N_4/N_3 = \mathrm{e}^{-\Delta E/kT}$$

so with a population of N_3 in level 3 there must also be a population $N_4 = N_3\mathrm{e}^{-\Delta E/kT}$ in level 4. Hence the absorbed pump energy is

$$W = (N_3 + N_4)V_g(hc/\lambda_p) = V_g(hc/\lambda_p)N_3\left(1 + \mathrm{e}^{-\Delta E/kT}\right)$$

which can be rewritten as

$$W = V_g(hc/\lambda_p)N_4\mathrm{e}^{\Delta E/kT}\left(1 + \mathrm{e}^{-\Delta E/kT}\right) = V_g(hc/\lambda_p)N_4\left(1 + \mathrm{e}^{\Delta E/kT}\right).$$

Pump energy [7 marks]

For transition α we have $N^* = N_3 - N_1$. Assuming all ions are in levels 1, 3 and 4 (since the other levels populated by optical pumping decay rapidly to the upper laser levels and we can neglect emission from level 4 as there is no feedback on the transition) we can write $N_T = N_1 + N_3 + N_4$. Thus

$$\begin{aligned} N^* &= N_3 - N_1 \\ &= N_3 - (N_T - N_3 - N_4) \\ &= 2N_3 + N_4 - N_T \\ &= \left(2 + \mathrm{e}^{-\Delta E/kT}\right) N_3 - N_T \end{aligned}$$

which rearranges to

$$N_3 = \frac{N^* + N_T}{2 + \mathrm{e}^{-\Delta E/kT}}.$$

Laser oscillation requires that N^* reaches a threshold value set by the cavity losses. This will be small compared with N_T and hence

$$N_3 \approx \frac{N_T}{2 + \mathrm{e}^{-\Delta E/kT}}.$$

Putting everything together gives

$$W_{\mathrm{th}} = V_g \frac{hc}{\lambda_p} N_T \left(\frac{1 + \mathrm{e}^{-\Delta E/kT}}{2 + \mathrm{e}^{-\Delta E/kT}} \right).$$

In the limit $T \to 0$ the exponentials tend to 0 and so

$$W_{\mathrm{th}} \to V_g \frac{hc}{\lambda_p} \frac{N_T}{2}$$

which is analogous to the ruby laser, for which the lasing threshold corresponds to promoting half the ions to the upper level. In the limit $T \to \infty$ the exponentials tend to 1 and

$$W_{\mathrm{th}} \to V_g \frac{hc}{\lambda_p} \frac{2N_T}{3}$$

so clearly the threshold energy increases with temperature.

Minimum power [5 marks]

For transition β we have $N^* = N_4 - N_2 \approx N_4$ as the lower level will be almost empty. For this four level laser the critical value of N_4 is set by the threshold condition

$$R_1 R_2 \mathrm{e}^{2N^* \sigma_{21} l_g} = 1 \quad \Rightarrow \quad 2N^* \sigma_{21} l_g = -\ln(R_1 R_2)$$

and so

$$N^* = \frac{-\ln(R_1 R_2)}{2\sigma_{21} l_g} \approx N_4.$$

Hence

$$W_{\mathrm{th}} = V_g \frac{hc}{\lambda_p} (N_3 + N_4) = \frac{-\ln(R_1 R_2)}{2\sigma_{21} l_g} V_g \frac{hc}{\lambda_p} \left(1 + \mathrm{e}^{\Delta E/kT}\right)$$

and using $V_g = \pi a^2 l_g$ gives the final result

$$W_{\mathrm{th}} = \frac{-\pi a^2}{2\sigma_{21}} \ln(R_1 R_2) \frac{hc}{\lambda_p} \left(1 + \mathrm{e}^{\Delta E/kT}\right)$$

Finally inserting the given values ($\lambda_p = 500\,\mathrm{nm}$, $R_1 = 1$, $R_2 = 0.9$, $a = 5\,\mathrm{mm}$, $\sigma_{21} = 7 \times 10^{-19}\,\mathrm{cm}^2$, $T = 333\,\mathrm{K}$ and $\Delta E = 800\,\mathrm{cm}^{-1}$) leads to $W_{\mathrm{th}} = 0.768\,\mathrm{J}$. The pump power is given by $P_{\mathrm{th}} = W_{\mathrm{th}}/\tau_2$, and as $\tau_2 = 260\,\mu\mathrm{s}$ we get $P_{\mathrm{th}} = 2.95\,\mathrm{kW}$.

Question 3: Interferometry

Introduction [6 marks]

The small tilt adds an additional optical path to each arm:

$$\begin{aligned} \delta &= L/\cos\psi - L \\ &\approx L[(1-\psi^2/2)^{-1} - 1] \\ &\approx L[(1+\psi^2/2) - 1] \\ &= L\psi^2/2. \end{aligned}$$

The clockwise path travels along 3 tilted sides, while the anti-clockwise path only travels along 1 tilted side. Therefore the total extra path length is 2δ, and the phase accumulated is $2\phi = (2\pi/\lambda)L\psi^2$.

Translating the mirror has no effect because the interferometer is a *common path* (or *Sagnac*) interferometer. The path length difference between clockwise and anti-clockwise paths is unaffected by translation since the path lengths for both paths change by the same amount.

Varying θ [9 marks]

The beamsplitter divides the input field into two modes, one propagating clockwise (c) and one anti-clockwise (a). Writing the amplitudes for these two modes as the components of a column vector, the field after the beamsplitter is

$$\begin{pmatrix} E_{\text{in}}/\sqrt{2} \\ \mathrm{i}E_{\text{in}}/\sqrt{2} \end{pmatrix}.$$

after reflection from M, each mode receives an equal and opposite transverse momentum kick k_x, where $\sin\theta = k_x/k$ with $k = 2\pi/\lambda$. In addition, the c mode accumulates the phase 2ϕ with respect to the a mode. After one round trip of the interferometer we therefore have

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}(\phi+k_x x)}E_{\text{in}}/\sqrt{2} \\ \mathrm{i}\mathrm{e}^{-\mathrm{i}(\phi+k_x x)}E_{\text{in}}/\sqrt{2} \end{pmatrix}.$$

Finally, the fields are recombined at the beamsplitter. The output port O is the 'dark port' of the interferometer. The output field is therefore given by

$$E_{\text{out}} = -\mathrm{i}\mathrm{e}^{\mathrm{i}(\phi+k_x x)}E_{\text{in}}/2 + \mathrm{i}\mathrm{e}^{-\mathrm{i}(\phi+k_x x)}E_{\text{in}}/2.$$

And some trivial algebra yields the result

$$E_{\text{out}}(x) = \sin(\phi + k_x x)E_{\text{in}}(x),$$

as required. Since $k_x x$, ϕ are small we have $\sin(\phi + k_x x) \approx \phi + k_x x$.

Gaussian beam [5 marks]

The input field Gaussian profile can be expanded as $e^{-(x/\sigma)^2} \approx 1 - x^2/\sigma^2$. Multiplying these two together yields

$$\begin{aligned} E_{\text{out}}(x) &\approx (\phi + k_x x)E_0(1 - x^2/\sigma^2) \\ &= \phi E_0(1 - x^2/\sigma^2 + k_x x/\phi - k_x x^2/\sigma^2 + \ldots) \\ &\approx \phi E_0 \left[1 - \left(\frac{x - k_x\sigma^2/2\phi}{\sigma}\right)^2 + \ldots\right] \\ &\approx \phi E_0 e^{-[(x - \langle x \rangle)/\sigma]^2}, \end{aligned} \tag{1}$$

where E_0 denotes the amplitude of the input field.

Small rotations [5 marks]

The displacement $\langle x \rangle$ depends *inversely* on ϕ. By choosing ϕ to be sufficiently small, any tiny rotation θ can be 'amplified' to give a measurable displacement. Very little light emerges from the output O. As just shown, the total intensity is proportional to $\phi^2|E_0|^2$. Therefore to maintain a measurable signal as ϕ is reduced, more and more laser power is required. Eventually the cost of a high power laser becomes prohibitive and the damage thresholds of the various components may be reached.

Question 4: Non-linear optics

Examiner's note: this question is slightly ambiguous about conventions for describing rotations, and any consistent choice is therefore acceptable. Note also that the assignment of marks on the examination paper is not quite correct, as the second last paragraph should be included in the [10] mark prism section.

Introduction [3 marks]

For a transparent crystal to display the Pockels effect it has to have a non-zero $\chi^{(2)}$; this in turn requires the crystal to be non-centrosymmetric.

If a crystal possesses inversion symmetry the application of an electric field E along some direction must cause a change in the refractive index $\Delta n = sE$. If the direction of the field is reversed the change becomes $\Delta n = s(-E)$, but inversion symmetry requires the two directions to be physically equivalent, and so $s = -s$ which requires $s = 0$. Thus linear Pockels crystals *cannot* have a centre of symmetry.

Indicatrix [8 marks]

A field applied in the z-direction induces a change in the refractive index in the xy-plane so that the index ellipsoid becomes

$$\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} + 2r_{63}xyE_z^0 = 1$$

A rotation of the x- and y-axes about the z-axis then transforms the ellipsoid equation into

$$\left(\frac{1}{n_o^2} - r_{63}E_z^0\right)x'^2 + \left(\frac{1}{n_o^2} + r_{63}E_z^0\right)y'^2 + \frac{z^2}{n_e^2} + 2r_{63}xyE_z^0 = 1$$

identifying

$$\frac{1}{n_{x'}^2} = \frac{1 - n_o^2 r_{63} E_z^0}{n_o^2}$$

and so

$$n_{x'} = n_o \left(1 - n_o^2 r_{63} E_z^0\right)^{-1/2} \approx n_o - \tfrac{1}{2} n_o^2 r_{63} E_z^0$$

where the approximation applies when $n_o^2 r_{63} E_z^0 \ll 1$. There is a similar result for $n_{y'}$ and we have for the difference in refractive indices along these rotated axes

$$\Delta n = |n_{x'} - n_{y'}| = n_o^2 r_{63} E_z^0.$$

Of the linearly polarised laser beam has the plane of polarisation inclined at 45° to the x'- and y'-axes and the electric field along the z-direction has the right magnitude such that it produces a π phase shift then the plane of polarisation will be rotated by 90°. The rotation angle

$$\phi = \frac{2\pi}{\lambda_0} \Delta n d = \pi \quad \Rightarrow \quad \Delta n = \frac{\lambda_0}{2d} = n_o^2 r_{63} E_z^0$$

occurs for a voltage V_0 given by

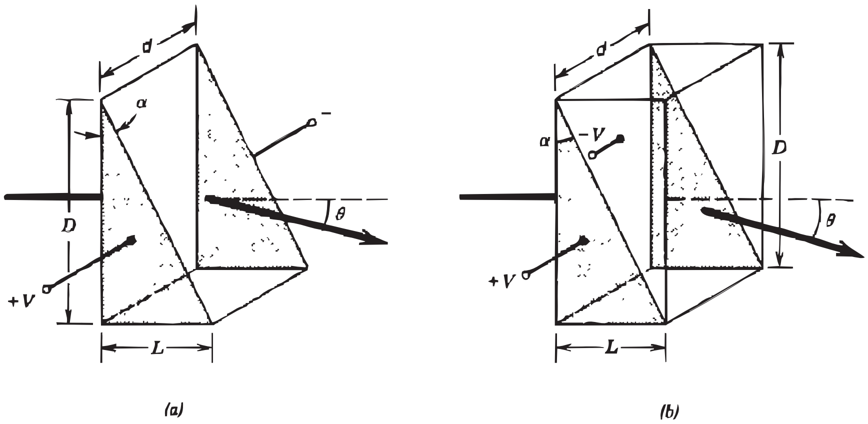
$$V_0 = dE_z^0 = \frac{\lambda_0}{2 n_o^3 r_{63}}.$$

Prism [10 marks]

The angle of deflection produced by a prism with a small apex angle α is given by $\theta \approx (n-1)\alpha$. If the light is linearly polarised along one of the rotated axes (say, x') and propagating along the other (y') then the electric field will cause a *change* in the deflection angle of

$$\Delta\theta = \alpha \Delta n = \left| \frac{1}{2} \alpha r_{63} n_o^3 E_z^0 \right| = \left| \frac{1}{2} \alpha r_{63} n_o^3 E_z^0 \frac{V_0}{d} \right|.$$

The second prism, which is identical to the first can be inverted with the two hypotenuses in near contact as shown below (sketch would be fine!)



An electric field reversed with respect to the first then doubles the electro-optic deflection while canceling the static crystallographic deflection.

A change in polarisation axis so that it is parallel with the z-axis leads to no electro-optic effect and the static refractive index is now the extraordinary index.

Practicalities [4 marks]

The degree to which the deflection can be detected depends on the diffraction of the beam; of the beam fills the input face of the double prism then

$$\delta\theta \approx \frac{\lambda_0}{D}$$

with $D \approx d$. Thus while making the aperture greater helps to reduce the diffraction spread, making deflection measurements easier, it also necessitates the application of higher voltages to produce the same electro-optic deflection.