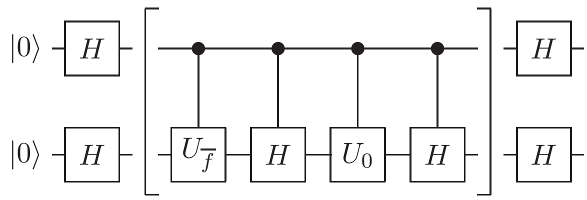
5. Grover's quantum search algorithm enables an efficient analysis of a binary function f from n bits to 1 bit by locating one of k satisfying inputs x for which $f(x) = 1$. A closely related algorithm, approximate quantum counting, permits the value of k to be estimated.

A quantum network that implements this algorithm in the case $n = 1$ is shown below. The central group of unitary operations, which is drawn surrounded by square brackets, is applied r times, and the value of the top qubit can be measured as a function of r. H indicates a Hadamard gate, U_0 maps $|0\rangle$ to $-|0\rangle$ and leaves $|1\rangle$ unchanged, and $U_{\overline{f}}$ maps $|x\rangle$ to $(-1)^{f(x)\oplus 1}|x\rangle$.



For the case $n = 1$, describe the four possible functions f and classify them according to their values of k. Give explicit matrix forms for U_0 and the four unitary operations $U_{\overline{f}}$. Explain why the controlled-Hadamard gates could be replaced by simple Hadamard gates applied to the second qubit. [8]

Find the state of the two qubits at the end of the calculation for each of the four functions in the case $r = 1$, and evaluate the expectation value of σ_z for the first qubit in each case. [10]

Now consider the situation with $r > 1$. For the function with $k = 0$ show that the first qubit always ends in the state $|0\rangle$, while for the function with $k = 2$ the qubit ends in the state $|r \bmod 2\rangle$. [7]