

## CP4: Optics

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### Part 2: Wave Optics

## Wave Optics

- In part 1 we saw how waves can appear to move in straight lines and so can explain the world of geometrical optics
- In part 2 we explore phenomena where the wave nature is obvious not hidden
- Key words are *interference* and *diffraction*

## Wave motion (1)

- See the waves lecture course for details!
- Basic form of a one-dimensional wave is  $\cos(kx - \omega t - \phi)$  where  $k = 2\pi/\lambda$  is the *wave number*,  $\omega = 2\pi\nu$  is the *angular frequency*, and  $\phi$  is the *phase*
- Various other conventions in use

## Wave motion (2)

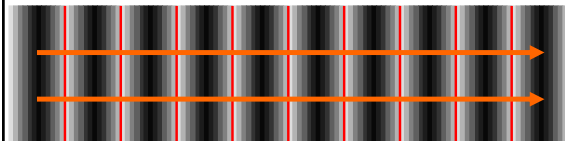
- Basic wave is  $\cos(kx - \omega t - \phi)$
- At time  $t$  the wave will look identical to its appearance at time 0 except that it will have moved forward by a distance  $x = \omega t/k$
- Wave moving the other way is described by  $\cos(-kx - \omega t - \phi)$

## Plane wave (1)

- This is the basic equation for a wave on a string, but we can also use the same approach to describe a light wave travelling along the x-axis
  - More complex versions for general motion
- Oscillations in the electric and magnetic fields which vary in space and time just like the motion of a string

## Plane wave (2)

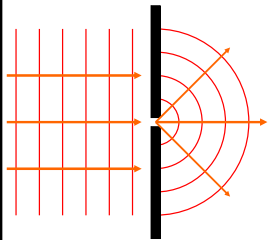
Plane wave is a series of parallel waves moving in one direction. Note that waves are really 3D: this is a 2D slice!



Mark the peaks of the waves as wavefronts

Draw normals to the wavefronts as rays

## Spherical wave

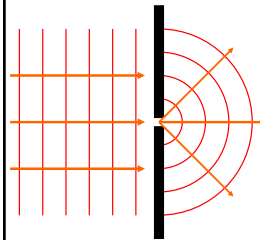


A plane wave impinging on a *infinitesimal* hole in a plate.

Acts as a point source and produces a *spherical* wave which spreads out in all directions.

Finite sized holes will be treated later.

## Cylindrical wave



A plane wave impinging on a *infinitesimal* slit in a plate.

Acts as a point source and produces a *cylindrical* wave which spreads out in all directions.

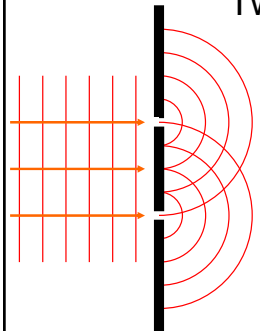
Finite sized slits will be treated later.

Remainder of this course will use slits unless specifically stated

## Obliquity Factor

- A more detailed treatment due to Fresnel shows that a point source does not really produce a spherical wave. Instead there is more light going forwards than sideways
  - Described by the *obliquity factor*  $K = \frac{1}{2}(1 + \cos\theta)$
  - Also explains lack of backwards wave
  - Not really important for small angles
  - Ignored in what follows

## Two slits

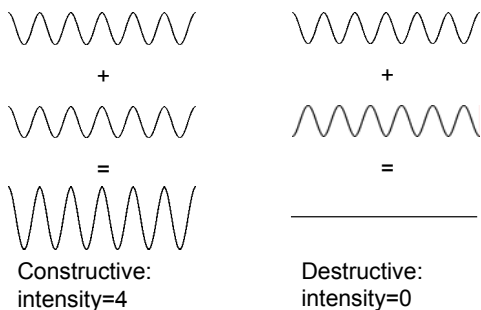


A plane wave impinging on a pair of slits in a plate will produce two circular waves

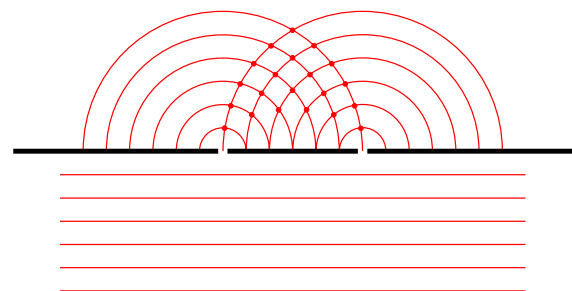
Where these overlap the waves will interfere with one another, either reinforcing or cancelling one another

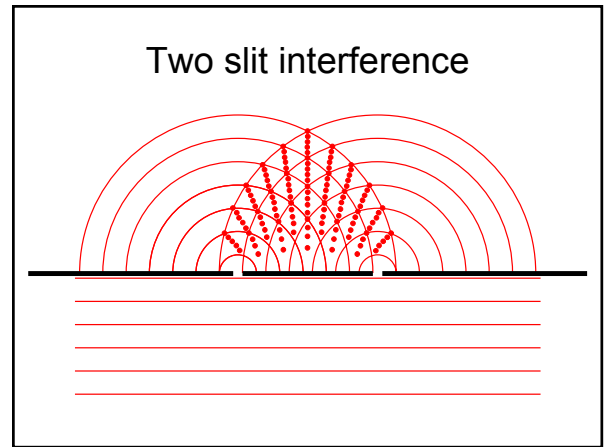
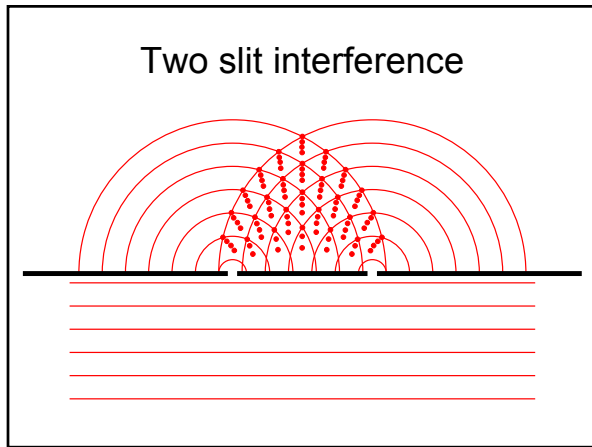
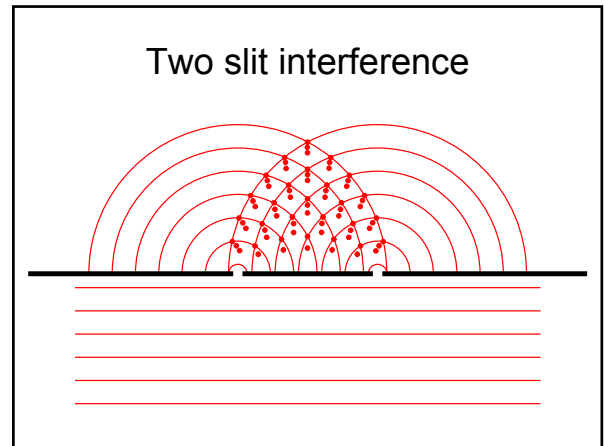
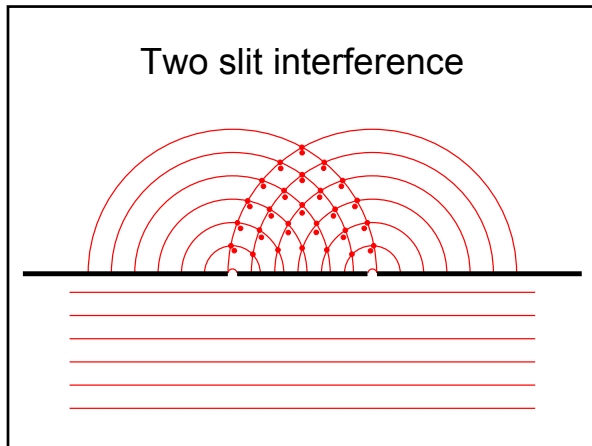
Intensity observed goes as square of the total amplitude

## Interference



## Two slit interference





### Two slit interference

Peaks in light intensity in certain directions (reinforcement)

Minima in light intensity in other directions (cancellation)

Can we calculate these directions directly without all this tedious drawing?

- ### Two slit interference (2)
- These lines are the sets of positions at which the waves from the two slits are *in phase* with one another
  - This means that the *optical path lengths* from the two slits to points on the line must differ by an integer number of wavelengths
  - The amplitude at a given point will oscillate with time (not interesting so ignore it!)

### Two slit interference (3)

Two path lengths are the same

Central line is locus of points at same distance from the two slits

### Two slit interference (4)

Path lengths related by  $x_{\text{left}} = x_{\text{right}} + \lambda$

Next line is locus of points where distance from the two slits differs by one wavelength

### Two slit interference (5)

Path lengths related by  $x_{\text{left}} = x_{\text{right}} + \lambda$

Next line is locus of points where distance from the two slits differs by one wavelength

### Two slit interference (6)

Interference pattern observed on a distant screen

As  $y \ll D$  the three blue lines are effectively parallel and all make an angle  $\theta \approx y/D$  to the normal. The bottom line is slightly longer than the top.

### Two slit interference (7)

Close up

Green line is normal to the blue lines, and forms a right angled triangle with smallest angle  $\theta$ . Its shortest side is the extra path length, and is of size  $d \times \sin(\theta) \approx dy/D$

### Two slit interference (8)

View at the screen

Central fringe midway between slits

Bright fringes seen when the extra path length is an integer number of wavelengths, so  $y = n\lambda D/d$

Dark fringes seen when  $y = (n + 1/2)\lambda D/d$

Taking  $\lambda = 500\text{nm}$ ,  $d = 1\text{mm}$ ,  $D = 1\text{m}$ , gives a fringe separation of  $0.5\text{mm}$

## Lloyd's Mirror

- There are a very large number of similar two-source interference experiments
- Lloyd's Mirror features on the practical course!
- Uses the interference between light from a slit and its virtual image in a plane mirror
  - See also Fresnel's mirror, Fresnel's biprism

## Practicalities: the screen

- Young's slits forms an interference pattern on a screen at *any* distance
  - Not an image!
  - At large distances the interference points lie on straight lines at constant angles
- Increasing the distance to the screen increases the separation between the fringes but decreases their brightness

## Practicalities: the slits

- The treatment above assumes that the slits act as point sources
  - Means that fringes can't be very bright
- As slits get broader the outer fringes blur
  - We will come back to this once we have looked at single slit diffraction
  - Not a problem as long as slit width is much smaller than slit separation
  - Allows central fringe to be identified!

## Practicalities: the source (1)

- Young's slits assumes a light source providing uniform plane wave illumination
- Simplest approach is to assume a point source far from the slits
  - Spherical waves look like plane waves at a long distance from the source
- Such sources are not very bright

## Practicalities: the source (2)

- A plane wave corresponds to a set of parallel rays
- This can be achieved by placing a point source at the focus point of a converging lens
- Can do a similar trick to see the fringes more clearly
  - More of this when we look at gratings

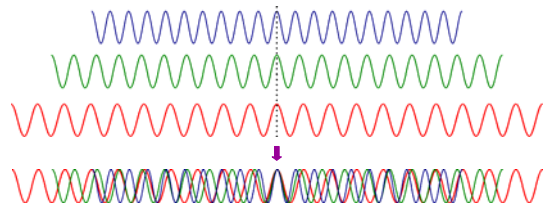
## Practicalities: the source (3)

- Still necessary to use a point source to avoid interference between light coming from different parts of the source!
- Detailed calculation is similar to that for slit size, so the source only needs to be small rather than a true point.
- Also turns out you can use a source *slit* as long as it is parallel to the two slits

### Practicalities: colours (1)

- The above all assumes a *monochromatic* light source
- Light of different colours does not interfere and so each colour creates its own fringes
- Fringe separation is proportional to wavelength and so red fringes are bigger than blue fringes
- Central fringe coincides in all cases

### Practicalities: colours (2)



Observe bright central fringe (white with coloured edges) surrounded by a complex pattern of colours. Makes central fringe easy to identify!

### Interference (1)

- It is easy to calculate the positions of maxima and minima, but what happens between them?
- Explicitly sum the amplitudes of the waves  
 $A = \cos(kx_1 - \omega t - \phi) + \cos(kx_2 - \omega t - \phi)$
- Write  $x_1 = x - \delta/2$ ,  $x_2 = x + \delta/2$  and use  
 $\cos(P+Q) + \cos(P-Q) = 2\cos(P)\cos(Q)$
- Simplifies to  $A = 2\cos(kx - \omega t - \phi) \times \cos(k\delta/2)$

### Interference (2)

- Amplitude is  $A = 2\cos(kx - \omega t - \phi) \times \cos(k\delta/2)$
- Intensity goes as square of amplitude so  
 $I = 4\cos^2(kx - \omega t - \phi) \times \cos^2(k\delta/2)$
- First term is a rapid oscillation at the frequency of the light; all the interest is in the second term
- $I = \cos^2(k\delta/2) = [1 + \cos(k\delta)]/2$

### Interference (3)

- Intensity is  $I = \cos^2(k\delta/2) = [1 + \cos(k\delta)]/2$
- Intensity oscillates with maxima at  $k\delta = 2n\pi$  and minima at  $k\delta = (2n+1)\pi$
- Path length difference is  $\delta = dy/D$
- Maxima at  $y = D\delta/d$  with  $\delta = 2n\pi/k$  and  $k = 2\pi/\lambda$  giving  $y = n\lambda D/d$

### Exponential waves (1)

- Whenever you see a cosine you should consider converting it to an exponential!  
 $\exp(ix) = \cos(x) + i \sin(x)$
- Basic wave in exponential form is  
 $\cos[i(kx - \omega t - \phi)] = \text{Re}\{\exp[i(kx - \omega t - \phi)]\}$
- Do the calculations in exponential form and convert back to trig functions at the very end

## Exponential waves (2)

- Repeat the interference calculation
- Explicitly sum the amplitudes of the waves
 
$$A = \text{Re} \{ \exp[i(kx - k\delta/2 - \omega t - \phi)] + \exp[i(kx + k\delta/2 - \omega t - \phi)] \}$$

$$= \text{Re} \{ \exp[i(kx - \omega t - \phi)] \times (\exp[-ik\delta/2] + \exp[+ik\delta/2]) \}$$

$$= \text{Re} \{ \exp[i(kx - \omega t - \phi)] \times 2\cos[k\delta/2] \}$$

$$= 2\cos(kx - \omega t - \phi) \times \cos[k\delta/2]$$
- Same result as before (of course!) but can be a bit simpler to calculate
- Will use complex waves where convenient from now on

## Exponential waves (3)

- We can do the sum in a slightly different way
 
$$A = \text{Re} \{ \exp[i(kx - \omega t - \phi)] + \exp[i(kx + k\delta - \omega t - \phi)] \}$$

$$= \text{Re} \{ \exp[i(kx - \omega t - \phi)] \times (1 + \exp[+ik\delta]) \}$$

$$= \text{Re} \{ \exp[i(kx - \omega t - \phi)] \times \exp[+ik\delta/2] \times (\exp[-ik\delta/2] + \exp[+ik\delta/2]) \}$$

$$= \text{Re} \{ \exp[i(kx - \omega t - \phi)] \times \exp[+ik\delta/2] \times 2\cos[k\delta/2] \}$$
- Taking the real part now looks messy because of the middle *phase* term, but we can just wait a time  $\tau$  such that  $\omega\tau = k\delta/2$  and everything comes back into phase. What really matters is the *absolute value* of the wave.

## Exponential waves (4)

- Use a complex amplitude to represent the wave
 
$$A = \exp[i(kx - \omega t - \phi)] \times \exp[+ik\delta/2] \times 2\cos[k\delta/2]$$
- The intensity of the light is then given by the square modulus of the amplitude:
 
$$I = A^*A = 4\cos^2[k\delta/2]$$
- This approach loses the rapid time oscillations, but we have previously ignored these anyway! Result is the peak intensity which is twice the average intensity.

## Phasors (1)

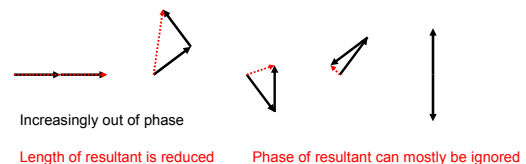
- Something very similar occurs in circuit theory
- We represent an oscillation by a complex function, which basically works but we have to fiddle a few results at the end
- A better approach is to use *phasors* which are mathematical objects which are almost but not quite identical to complex numbers. See Lorrain and Corson for the gory details.
- Phasors in optics are similar (but not quite the same!)

## Phasors (2)

- A complex wave has an amplitude, an oscillatory part, and a phase  $\psi = A \times \exp[i(kx - \omega t)] \times \exp[i\phi]$
- Note that the phase term will depend on things like paths lengths measured in multiples of the wavelength
- The oscillatory bit is not terribly interesting, and we can combine the amplitude and phase to get the complex amplitude  $\alpha = A \times \exp[i\phi]$
- Interference is about adding complex amplitudes

## Phasors (3)

- We can represent a complex amplitude as a two dimensional vector on an Argand diagram
- We can then get the sum of complex amplitudes by taking the vector sum



## Phasors (4)

- In some cases elegant geometrical methods can be used to say something about the sum of a set of phasors without doing tedious calculations
- No calculations actually require phasors to be used, and at this level they are not terribly useful
- Don't worry about them too much!

## Optical path lengths (1)

- We can change the appearance of a two slit interference pattern by changing the optical path length for light from one of the two slits
- Place a piece of transparent material with refractive index  $n$  and thickness  $w$  in front of one of the slits
- This increases the optical path length for light travelling through that slit by  $(n-1) \times w$ , changing the phase of the light

## Optical path lengths (2)

- We could recalculate everything from first principles, but it is simplest just to note that the *central fringe* corresponds to the point where the *optical path lengths* for the two sources are identical
- Light travelling through the transparent material has travelled a *longer* optical path and so the central fringe must move *towards* this slit to compensate

## Optical path lengths (3)



Simple case: waves arrive at the slits in phase and the central interference peak is exactly between the slits

## Optical path lengths (4)



One slit covered: waves arrive at the upper slit delayed by half a wavelength and the central interference peak is moved up by half a peak

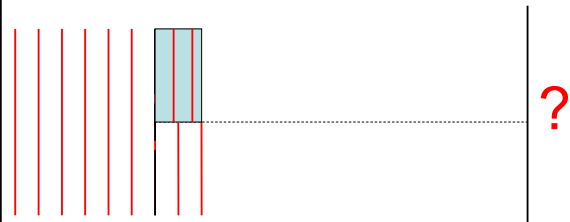
## Recognising the central peak

- Taking  $\lambda=500\text{nm}$ , a piece of glass  $0.1\text{mm}$  thick with  $n=1.5$  gives a shift of 100 fringes
- The above all assumes that we can *recognise* the central peak, but in the naïve treatment all peaks look the same!
- For white light fringes the central peak is easily recognised as the only clear white fringe
- For monochromatic light imperfections (notably the finite slit width) means that the central peak will be the brightest

### Optical path lengths (5)

- It might seem odd that we always talk about the transparent material being placed *before* the slit rather than after it
- This means that we have to do the optical path calculations from some apparently arbitrary point before the slit
- Why not place the transparent material after the slit?

### Optical path lengths (6)



Situation superficially similar to the previous case, but in fact the light will travel through the transparent medium at an angle making calculations messy!

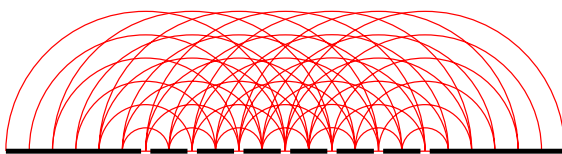
### Optical path lengths (7)

- What about the apparently arbitrary start point?
- Rigorous approach is to start all calculations from the *source*, not from near the slits
- Also allows calculations on the effect of moving the source nearer to one of the two slits!
- This is also how you should think about imperfections like finite source sizes: different parts of the source lie at different distances from the two slits.

### Diffraction gratings (1)

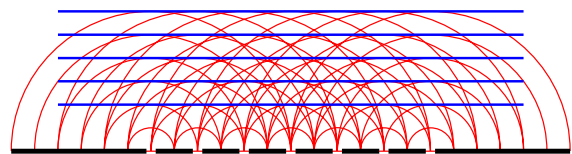
- A diffraction grating is an extension of a double slit experiment to a very large number of slits
- Gratings can work in transmission or reflection but we will only consider transmission gratings
- The basic properties are easily understood from simple sketches, and most of the advanced properties are (in principle!) off-syllabus

### Diffraction gratings (2)



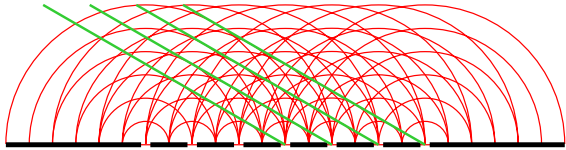
Circular waves are formed from each source

### Diffraction gratings (3)



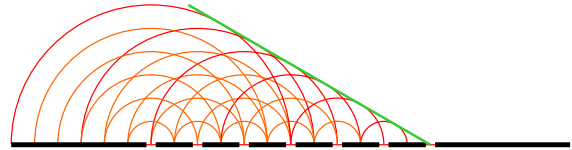
Huygens style reinforcement creates a forward wave

### Diffraction gratings (4)



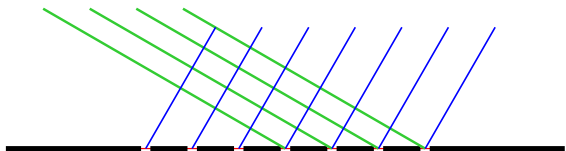
Also get reinforcement at angles!

### Diffraction gratings (5)



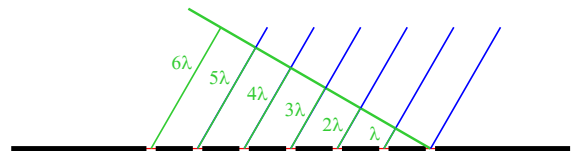
Reinforcement comes from successive waves from neighbouring sources

### Diffraction gratings (6)



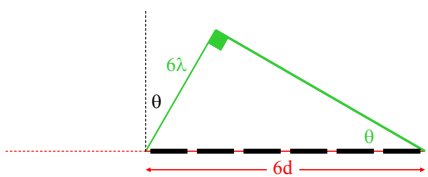
Draw rays normal to the wavefronts

### Diffraction gratings (7)



Points on a wavefront must be in phase, so the extra distances travelled must be multiples of a wavelength

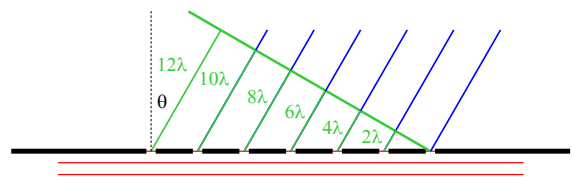
### Diffraction gratings (8)



From trigonometry we see that  $\sin(\theta) = 6\lambda / 6d$  where  $\theta$  is the angle between the ray direction and the normal

Basic diffraction equation:  $\lambda = d \sin(\theta)$

### Diffraction gratings (9)

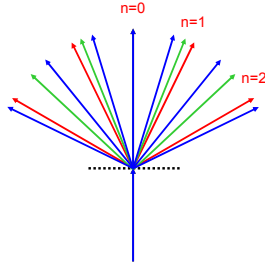


Can also get diffraction in the same direction if the wavelength is halved!

General diffraction equation:  $n \lambda = d \sin(\theta)$

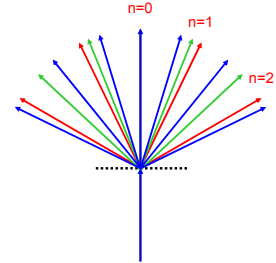
## Dispersion (1)

- $n\lambda = d \sin(\theta)$
- Different colours have different wavelengths and so diffract at different angles
- Take red ( $\lambda=650\text{nm}$ ), green ( $\lambda=550\text{nm}$ ), and blue ( $\lambda=450\text{nm}$ ) light and  $d=1.5\mu\text{m}$

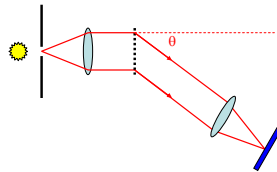


## Dispersion (2)

- Angular dispersion of a grating is  $D = d\theta/d\lambda$   
 $= n/d \cos(\theta)$   
 $= n/\sqrt{d^2 - n^2\lambda^2}$
- Increases with *order* of the spectrum and when  $d \approx \lambda$
- Note that high order spectra can overlap!



## Grating Spectrometer



Light from a source is passed through a narrow source, collimated with a lens, dispersed with a grating, focused with a lens, and then detected

Measure intensity as a function of  $\theta$  to get the spectrum (in practice it is better to measure  $2\theta$ , the angle *between* lines). Real designs more complex!

## Prisms or gratings?

- Gratings have many advantages
  - Dispersion can be calculated!
  - High orders lead to high resolution
  - Reflection gratings don't need transparency
- Gratings have a few disadvantages
  - Intensity shared between different orders
  - *Can be improved by blazing the grating*

## “Natural” diffraction gratings



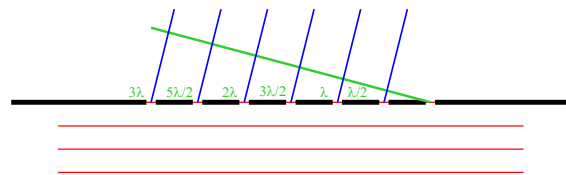
Wings of butterflies (Blue Morpho)

Opals



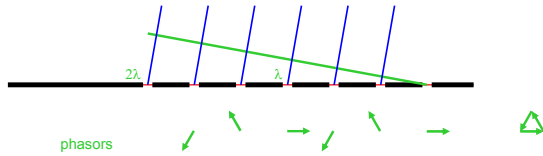
Compact discs

## Between the peaks (1)



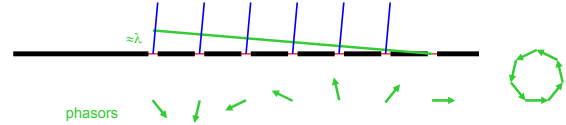
We also need to work out what happens in other directions. Between  $n=0$  and  $n=1$  there is a direction where the light from adjacent sources exactly cancel each other out – just as for two slits!

### Between the peaks (2)



Closer to  $n=0$  there is a direction where light from groups of three adjacent sources exactly cancels out  
 With an *infinite* number of sources will get cancellation in *every* direction except for the main peaks!

### Between the peaks (3)



With an *finite* number of sources the last effective cancellation will occur when the path length difference between the first and last sources is about  $\lambda$ .

### Resolution of a grating

- The angular resolution of a diffraction grating can be calculated from the above. If the grating has  $N$  slits then need  $\lambda \approx Nd \sin \theta \approx Nd \theta$  which gives an angular resolution of  $\delta \theta \approx \lambda / Nd \approx \lambda / W$  where  $W$  is the width of the grating
- Similar results can be calculated in many different ways, most of which are rather complicated but one of which we will see

### Fraunhofer diffraction (1)

- In the above we mostly treated diffraction gratings as a generalisation of a double slit
- A more general approach is to use the theory of Fraunhofer diffraction
- This is applicable when the object is illuminated with plane waves and observed with plane waves
- **Otherwise have Fresnel diffraction (nasty!)**

### Fraunhofer diffraction (2)

- Fraunhofer diffraction applies when both the source and the image are a very long way from the diffracting object. Need distances greater than  $a^2/\lambda$  where  $a$  is the size of the object.
- More practically can use lenses to create and observe parallel beams as seen above for diffraction gratings

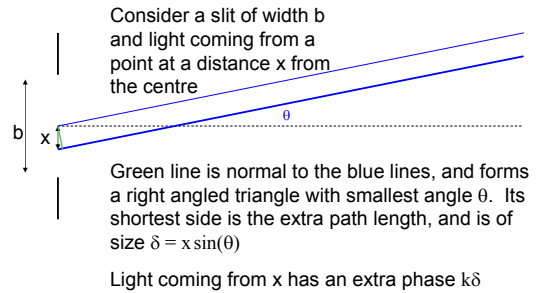
### Fraunhofer diffraction (3)

- Fraunhofer diffraction works by summing the amplitudes of light waves coming from all sources. These interfere and the intensity is determined by the square of the total amplitude.
- For continuous objects need an integral
- Usually most convenient in this case to use the complex wave notation
- **This ultimately leads to the use of Fourier transforms in optics (next year!)**

### Single slit diffraction (1)

- Consider a single slit illuminated by a plane wave source
- If the slit is perfectly narrow this will create a cylindrical wave
- If the slit has significant width we must treat it as a continuous distribution of sources and integrate over them

### Single slit diffraction (2)



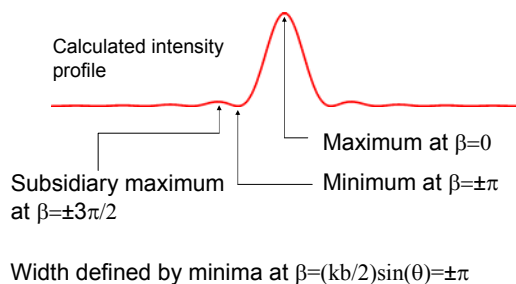
### Single slit diffraction (3)

- Final amplitude is obtained by integrating over all points  $x$  in the slit and combining the phases, most simply in complex form
- $A = \int \exp(ik\delta) dx = \int \exp(ikx \sin(\theta)) dx$
- Integral runs from  $x=-b/2$  to  $x=b/2$  and then normalise by dividing by  $b$

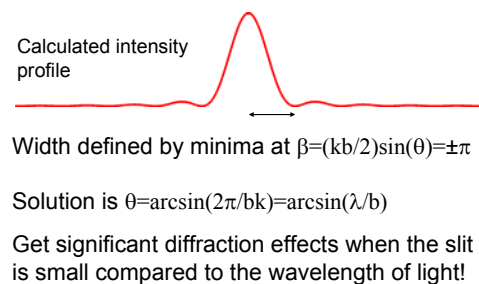
### Single slit diffraction (4)

- $A = \sin[(kb/2)\sin(\theta)] / [(kb/2)\sin(\theta)]$
- $A = \sin(\beta) / \beta$  with  $\beta = (kb/2)\sin(\theta)$
- Light intensity goes as the square of  $A$
- $I = I_0 [\sin(\beta) / \beta]^2$
- $I = I_0 \text{sinc}^2(\beta)$

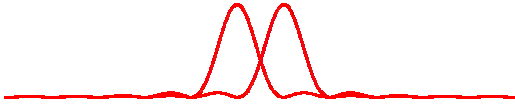
### Single slit diffraction (5)



### Single slit diffraction (6)



## Single slit diffraction (7)



The *Rayleigh Criterion* says that two diffraction limited images are well resolved when the maximum of one coincides with the first minimum of the other, so  $\theta_R = \arcsin(\lambda/b)$

## Resolution of a grating

- Treat a real grating of width  $W$  as an ideal (infinite) grating, followed by a slit of width  $W$  which blurs the pattern by single slit diffraction
- The resulting resolution is  $\delta\theta \approx \lambda/W$  exactly as calculated previously

## Slit width (two slits)

- The traditional treatment of the double slit experiment is that each narrow slit acts as a source of circular waves that interfere
- If the slits have finite width  $b$  they will only produce waves within an angle  $\theta = \arcsin(\lambda/b) \approx \lambda/b$
- Number of fringes seen is around  $D/b$  where  $D$  is the slit separation

## Resolution of a lens (1)

- Even neglecting aberrations a lens cannot give a perfect image
- Treat a real lens of width (diameter)  $W$  as an ideal (infinite) lens, followed by a circular hole of diameter  $W$  which blurs the image by diffraction
- The problem can be solved in much the same way as a single slit, but is a bit more complicated as we have to worry about two dimensions!
- The resulting resolution is  $\delta\theta \approx 1.22\lambda/W$

## Resolution of a lens (2)

- Thus a lens of focal length  $f$  can focus a beam down to a spot of diameter  $1.22f\lambda/W$  where  $W$  is the smaller of the width of the lens and the width of the light beam
- Note that resolution is better for blue (short wavelength) light than for red light
- There are corresponding limits on the ability of optical systems to produce parallel beams from point sources and on the ability of detectors to distinguish distant sources

## Resolution of a lens (3)

- The pupil of the human eye gives a limiting angular resolution around  $0.1\text{mrad}$  ( $20\text{arcsec}$ ). This corresponds to resolving the headlights on a car about  $20\text{km}$  away. With a small  $125\text{mm}$  telescope the resolution is about 20 times better!
- Wavelength is just as important as diameter: the Arecibo telescope ( $300\text{m}$  diameter) has a limiting angular resolution of about  $0.1\text{mrad}$  for  $3\text{cm}$  radiation, but only  $25\text{mrad}$  at  $6\text{m}$